Nonlinear elliptic equations in unbounded cylinder via inversions

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Introduction

- PDEs (e.g elliptic pdes) are used to model many problems in physics and the natural sciences. E.g. in electromagnetism, fluid dynamics, image analysis, among others.
- One of the physical problems e.g. in physics is the minimisation of energy using pde, e.g. Laplace equation

$$\Delta u = 0, \tag{1.1}$$

where $u : \Omega \to \mathbf{R}$ defined on a domain $\Omega \subset \mathbf{R}^n$, which is equivalent to

$$\min \int |\nabla u|^2 dx.$$

- Solutions to (1.1) describe for example gravitational (and electrostatic) potentials of force fields, temperature distributions in some medium.

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- However, many problems, e.g non-Newtonian fluids such as paint, glaciers and molten plastics are highly nonlinear.
- Thus, nonlinear cousin to (1.1), the so-called p-Laplace equation

$$\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad (1.2)$$

where $1 < p < \infty$. Equation (1.2) is the Euler-Lagrange equation minimizing the p-energy

$$\int |\nabla u|^p \, dx$$

among functions $u = f$ on $\partial\Omega$.

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- Not possible to find Solutions to (1.2) in C^2 in classical sense.
- But rather (1.2) is understood in a weak sense by means of Sobolev spaces.

Definition

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a weak solution of the p-Laplace equation if for all test functions $\varphi \in C_0^\infty(\Omega)$, then

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = 0, \tag{1.3}$$

where Ω is an open subset of Euclidean space $\mathbf{R}^n$, $n \geq 2$.

Cont'd

- To establish the existence of weak solutions, we study their properties such as uniqueness and regularity.
- Celebrated Wiener criterion in terms of sobolev capacities is applied.

Definition

A point x_0 at the boundary $\partial\Omega$ of the domain Ω in $\mathbf{R}^n$ is regular if for any boundary data given by a continuous function $f \in \mathcal{C}(\partial\Omega)$ and for the solution u of a Dirichlet problem constructed from f,

$$\lim_{x \to x_0} u(x) = f(x_0),$$

holds.

Motivation/Results

- Björn [1] considered a mixed BVP

$$\operatorname{div}(\mathcal{A}(x)\nabla u(x)) = 0 \tag{2.1}$$

  in G_0 with $\mathcal{A}(x)$ is symmetric matrix
- The Wiener regularity properties in terms of Sobolev capacities for weak solutions at infinity were obtained. The results were for a second order non-uniform elliptic equation (2.1).
- In our case, we are considering (1.2)- nonlinear case to establish the regularity of weak solutions at infinity.

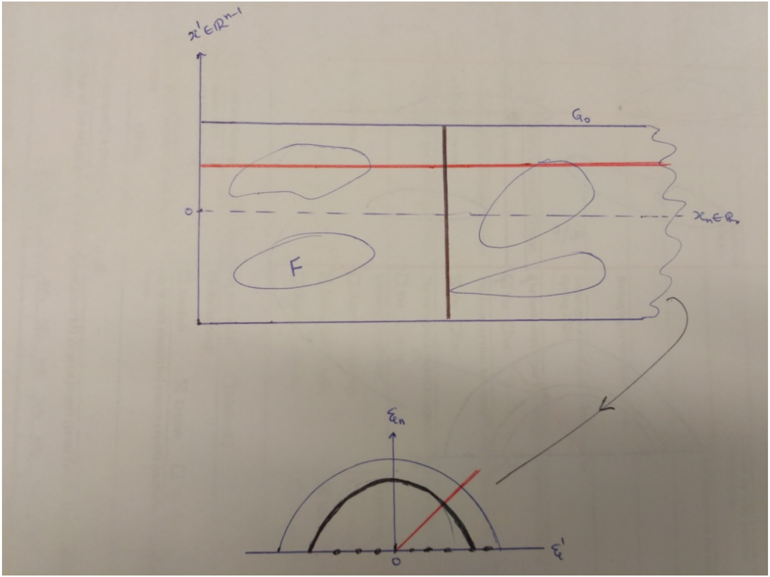
Motivation/results

- Consider $G_0 = \omega \times (0, \infty)$ be an infinite half cylinder in $\mathbf{R}^n$. Let F be a closed unbounded subset of $\bar{G}_0$ such that $G_0 \backslash F$ is connected.
- In $G_0 \setminus F$, consider (1.2), $x \in \bar{G}_0 \setminus F$ with boundary conditions $u = 0$ on F (Dirichlet) and $\frac{\partial u}{\partial n} = 0$ on $\partial G_0 \setminus F$ (Neumann)
- Let $T : G_0 \to B^+(0,1)$ be the mapping given by

$$\xi' = \frac{2e^{-\kappa x_n} x'}{1 + |x'|^2}, \quad \xi_n = \frac{e^{-\kappa x_n}(1 - |x'|^2)}{1 + |x'|^2},$$

where $x = (x', x_n) \in G_0$ and $\xi = (\xi', \xi_n) \in T(G_0)$.

Motivation/Results



Motivation/Results

- T is C^∞-diffeomorphism, then for every $x \in \mathbf{R}^n$ there is a bounded linear map $dT(x) : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T(x+h) - T(x) = dT(x)h + \mathcal{O}(h), \quad h \in \mathbf{R}^n.$$

- After this transformation, we get a new equation

$$\operatorname{div}_\xi \mathcal{B}(\xi, \nabla_\xi u) = 0 \tag{2.2}$$

  on $T(G_0)$ corresponds to (1.2) on G_0.
- $\mathcal{B}$ in (2.2) satisfies ellipticity condition with a weight w, where $w \in A_p$ class.

Results

Definition

A weight w is in Muckenhoupt's A_p class, if w is a nonnegatively locally integrable functions in $\mathbf{R}^n$ such that

$$\sup \left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{\frac{1}{(1-p)}}\, dx \right)^{p-1} = C_{w,p} < \infty,$$

w is identified with measure μ as $\mu = \int_E w(x)dx$.

- The operator $\mathcal{B}$ also satisfy boundedness and monotonicity conditions.

Methods and tools

- Quasiregular mapping was applied to transform G_0 into a suitable bounded domain (unit ball in $\mathbf{R}^n$). -boundary data is preserved.
- Suitable function spaces on G_0 and $T(G_0)$ have been stated.
- With the help of Wiener criterion results proved for nonlinear elliptic equations by Maz′ya [**?**], Kipeläinen-Malý [4] and Mikkonen [3], can the boundary regularity properties for weak solutions of (1.2) in G_0 be attained? It is what we are working on.

Impact and application

- The p-Laplace equation may describe the flow of non-Newtonian fluids like paint, blood, glaciers and molten plastics. For example, it can be used to design molds for plastic products or to predict the movement of glaciers.
- Understanding regularity of the solutions of such problems will solve many practical problems that involve p-Laplace operator.
- The study will also contribute to general understanding of regularity of weak solutions for p-Laplace equation on unbounded domains.
- Equip me with knowledge and research skills in the field of potential theory.

Reference

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Thank you!