Nonlinear elliptic equations in unbounded cylinder via inversions

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Second Network Meeting for Sida- and ISP-funded PhD Students in Mathematics
Stockholm 26–27 February 2018
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Order of presentation

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3. Impact and Application of my Research
PDEs (e.g. elliptic pdes) are used to model many problems in physics and the natural sciences. E.g. in electromagnetism, fluid dynamics, image analysis, among others.

One of the physical problems e.g. in physics is the minimisation of energy using pde, e.g. Laplace equation

$$\Delta u = 0,$$  \hspace{1cm} (1.1)

where $u : \Omega \rightarrow \mathbb{R}$ defined on a domain $\Omega \subset \mathbb{R}^n$, which is equivalent to

$$\min \int |\nabla u|^2 dx.$$ 

Solutions to (1.1) describe for example gravitational (and electrostatic) potentials of force fields, temperature distributions in some medium.
However, many problems, e.g. non-Newtonian fluids such as paint, glaciers and molten plastics are highly nonlinear.

Thus, nonlinear cousin to (1.1), the so-called $p$-Laplace equation

$$\Delta_p u := - \text{div}(|\nabla u|^{p-2} \nabla u) = 0,$$ (1.2)

where $1 < p < \infty$. Equation (1.2) is the Euler-Lagrange equation minimizing the $p$-energy

$$\int |\nabla u|^p \, dx$$

among functions $u = f$ on $\partial \Omega$. 
Not possible to find Solutions to (1.2) in $C^2$ in classical sense.
But rather (1.2) is understood in a weak sense by means of Sobolev spaces.

**Definition**

A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is said to be a weak solution of the $p$-Laplace equation if for all test functions $\varphi \in C_0^\infty(\Omega)$, then

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = 0,
$$

where $\Omega$ is an open subset of Euclidean space $\mathbb{R}^n$, $n \geq 2$. 
To establish the existence of weak solutions, we study their properties such as uniqueness and regularity.

Celebrated Wiener criterion in terms of Sobolev capacities is applied.

**Definition**

A point $x_0$ at the boundary $\partial \Omega$ of the domain $\Omega$ in $\mathbb{R}^n$ is regular if for any boundary data given by a continuous function $f \in C(\partial \Omega)$ and for the solution $u$ of a Dirichlet problem constructed from $f$, \[
\lim_{x \to x_0} u(x) = f(x_0),
\] holds.
Björn [1] considered a mixed BVP

\[
\text{div} (\mathcal{A}(x) \nabla u(x)) = 0
\]  

(2.1)
in \( G_0 \) with \( \mathcal{A}(x) \) is symmetric matrix

The Wiener regularity properties in terms of Sobolev capacities for weak solutions at infinity were obtained. The results were for a second order non-uniform elliptic equation (2.1).

In our case, we are considering (1.2)- nonlinear case to establish the regularity of weak solutions at infinity.
Consider $G_0 = \omega \times (0, \infty)$ be an infinite half cylinder in $\mathbb{R}^n$. Let $F$ be a closed unbounded subset of $\tilde{G}_0$ such that $G_0 \setminus F$ is connected.

In $G_0 \setminus F$, consider (1.2), $x \in \tilde{G}_0 \setminus F$ with boundary conditions $u = 0$ on $F$ (Dirichlet) and $\frac{\partial u}{\partial n} = 0$ on $\partial G_0 \setminus F$ (Neumann).

Let $T : G_0 \to B^+(0, 1)$ be the mapping given by

$$
\xi' = \frac{2e^{-\kappa x_n} x'}{1 + |x'|^2}, \quad \xi_n = \frac{e^{-\kappa x_n}(1 - |x'|^2)}{1 + |x'|^2},
$$

where $x = (x', x_n) \in G_0$ and $\xi = (\xi', \xi_n) \in T(G_0)$. 

Motivation/Results

- $T$ is $C^\infty$-diffeomorphism, then for every $x \in \mathbb{R}^n$ there is a bounded linear map $dT(x) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(x + h) - T(x) = dT(x)h + O(h), \quad h \in \mathbb{R}^n.$$ 

- After this transformation, we get a new equation

$$\text{div}_\xi B(\xi, \nabla_\xi u) = 0 \quad (2.2)$$

on $T(G_0)$ corresponds to (1.2) on $G_0$.

- $B$ in (2.2) satisfies ellipticity condition with a weight $w$, where $w \in A_p$ class.
Definition

A weight $w$ is in Muckenhoupt's $A_p$ class, if $w$ is a nonnegatively locally integrable functions in $\mathbb{R}^n$ such that

$$\sup \left( \int_B w(x) \, dx \right) \left( \int_B w(x)^{\frac{1}{1-p}} \, dx \right)^{p-1} = C_{w,p} < \infty,$$

$w$ is identified with measure $\mu$ as $\mu = \int_E w(x) \, dx$.

- The operator $B$ also satisfy boundedness and monotonicity conditions.
Methods and tools

- Quasiregular mapping was applied to transform $G_0$ into a suitable bounded domain (unit ball in $\mathbb{R}^n$). Boundary data is preserved.
- Suitable function spaces on $G_0$ and $T(G_0)$ have been stated.
- With the help of Wiener criterion results proved for nonlinear elliptic equations by Maz’ya [?], Kipeläinen-Malý [4] and Mikkonen [3], can the boundary regularity properties for weak solutions of (1.2) in $G_0$ be attained? It is what we are working on.
The $p$-Laplace equation may describe the flow of non-Newtonian fluids like paint, blood, glaciers and molten plastics. For example, it can be used to design molds for plastic products or to predict the movement of glaciers.

Understanding regularity of the solutions of such problems will solve many practical problems that involve $p$-Laplace operator.

The study will also contribute to general understanding of regularity of weak solutions for $p$-Laplace equation on unbounded domains.

Equip me with knowledge and research skills in the field of potential theory.


Thank you!