Coherent functors and asymptotic stability

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Second Network Meeting for Sida- and ISP-funded PhD Students in Mathematics

Stockholm 26–27 February 2018

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Introduction

- The topic of asymptotic prime divisors is a classic topic in commutative algebra.
- In 1979, Brodmann showed that when N is a finite module over the noetherian commutative ring A and $\mathfrak{a}$ is an ideal of A, then the sequence $\mathrm{Ass}_A(N/\mathfrak{a}^n N), n = 1, 2, ...$ of associated prime ideals is ultimately constant, that is there is m such that

$$\mathrm{Ass}_A(N/\mathfrak{a}^m N) = \mathrm{Ass}_A(N/\mathfrak{a}^{m+1} N) = \dots$$

  for all large m.
- This was generalized by Peter Schenzel and Lief Melkersson in 1993 to derived functors. There it was shown also that for every i, the sequence $Tor_i^A(A/\mathfrak{a}^n, N)$ eventually stabilizes.
- Also, in 1990, Leif Melkersson proved that $\mathrm{Ass}_{R_0}(M_n)$ stabilizes where $R = \oplus_{n=0}^{\infty} R_n$ is a noetherian homogeneous ring and $M = \oplus_{n=0}^{\infty} M_n$ is a finitely generated graded R-module.
- Similar results related to exact functors exist in literature.

Research

In this work, we want to prove results of asymptotic stability related to coherent functors. In particular, for this talk, we answer two questions:

1. Given a finitely generated module M and a coherent functor F. Does $\operatorname{Ass}_A F(M/M_n)$ stabilize?
2. Let R be a noetherian homogeneous ring and let M be a finitely generated graded R–module. If $A = R_0 \cong R/R_+$ and F is a coherent functor, does $\operatorname{Ass}_A F(M_n)$ stabilize?

Definitions

Let A be a ring, $\mathfrak{a}$ an ideal in A and M be an A-module.

1) The annihilator of an element x of M is the set $\mathrm{Ann}(x)$ of all $a \in A$ such that $ax = 0$. It is an ideal of A, the kernel of the map $A \to M$ sending 1 to x.
2) The annihilator of M is the set of all a in A such that $ax = 0$ for all x in M. It is also an ideal of R, and is the intersection of the ideals $\mathrm{Ann}(x) : x \in M$.
3) A prime ideal P is associated to M if there is some element x of M such that $\mathrm{Ann}(x) = P$. The set of such prime ideals is denoted by $\mathrm{Ass}_A(M)$.

Definitions

4) An (infinite) chain $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \ldots$ where the M_n are sub-modules of M is called a filtration of M, and denoted by (M_n). It is an $\mathfrak{a}$-filtration if $\mathfrak{a}M_n \subseteq M_{n+1}$ for all n, and a stable $\mathfrak{a}$-filtration if $\mathfrak{a}M = M_{n+1}$ for all sufficiently large n. Thus $(\mathfrak{a}^n M)$ is a stable $\mathfrak{a}$-filtration.

5) Let A be a noetherian ring and let $\mathcal{M}_A^{fg}$ be the category of finitely generated A–modules. A covariant functor $F : \mathcal{M}_A^{fg} \to \mathcal{M}_A^{fg}$ is said to be *coherent* if, for some morphism $f : M \to N$ of finitely generated A-modules, there is an exact sequence

$$Hom(N, -) \to Hom(M, -) \to F \to 0,$$

where $Hom(X, Y)$ is the set of all maps from X to Y.

Main results

We begin with

Lemma

Let $\{A(n)\}_{n=1}^{\infty}$ *and* $\{B(n)\}_{n=1}^{\infty}$ *be sequences of finite sets that satisfy*

$$B(n) \subset A(n+1) \subset A(n) \cup B(n)$$

for all n*. Assume that* $\{B(n)\}_{n=1}^{\infty}$ *is stable for sufficiently large* n*. Then the sequence* $\{A(n)\}_{n=1}^{\infty}$ *is ultimately constant.*

Proof.

Since $\{B(n)\}_{n=1}^{\infty}$ stabilizes, there exist and m such that $B(n) = B(m)$ for all $n \geq m$. Let x be in $A(n+1)$ where $n \geq m+1$. If $x \notin A(n)$, then $x \in B(n)$. Since by assumption $n-1 \geq m$, we have $x \in B(n-1) \subset A(n)$ a contradiction. Hence, $x \in A(n)$ for $n \geq m+1$. Since $A(n+1) \subset A(n)$ for all $n \geq m+1$ and these are finite sets, there is $l \geq m+1$ such that $A(l) = A(l+1) = \dots$ □

Main Results

Theorem

Let A be a noetherian ring, M a finitely generated A-module and F be a half exact coherent functor on the category of finitely generated A-modules. Then for any ideal $\mathfrak{a}$ in A and a stable $\mathfrak{a}$–filtration $\{M_n\}_{n\geq 0}$, we have that $\operatorname{Ass}_A F(M/M_n)$ is ultimately constant.

Proof

Let $G(A) = \oplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$ be the associated graded ring and let $G(M) = \oplus_{n \geq 0} M_n / M_{n+1}$. Then $G(M)$ is a finitely generated graded $G(A)$–module. Now consider the exact sequence of $R(\mathfrak{a})$–modules

$$0 \to G(M)_n \to M/M_{n+1} \to M/M_n \to 0$$

where $R(\mathfrak{a})$ is the Rees algebra. There is an exact sequence of $R(\mathfrak{a})$-modules

$$\oplus_{n \geq 0} F(G(M)_n) \to \oplus_{n \geq 0} F(M/M_{n+1}) \to \oplus_{n \geq 0} F(M/M_n)$$

with F coherent. Since $G(M)$ is a finitely generated $R(\mathfrak{a})$–module and $G(A) \cong R(\mathfrak{a})/\mathfrak{a}R(\mathfrak{a})$, we have that $\oplus_{n \geq 0} F(G(M)_n)$ is finitely generated $R(\mathfrak{a})$–module.

Proof cond

Let $N = \text{Im}(\oplus_{n\geq 0} F(G(M)_n) \to \oplus_{n\geq 0} F(M/M_{n+1})$, then N is a finitely generated graded $R(\mathfrak{a})$–module and we have an exact sequence of A–modules

$$0 \to N_n \to F(M/M_{n+1}) \to F(M/M_n).$$

Now, $\operatorname{Ass} N_n$ stabilizes for large n so by the lemma above, we get that $\operatorname{Ass} F(M/M_{n+1}) \subset \operatorname{Ass} F(M/M_n)$ for n large enough.

Corollary

Let A be a noetherian ring, $\mathfrak{a}$ an ideal in A and let F be a half exact coherent functor on the category of finitely generated A-modules. Then $\operatorname{Ass} F(A/\mathfrak{a}^n)$ stabilizes for large n.

Theorem

Let $R = \oplus_{n \geq 0} R_n$ be a noetheriam homogeneous ring and let $M = \oplus_{n \geq 0} M_n$ be a finitely generated graded R–module. Let $A = R_0 \cong R/R_+$ and let F be a coherent functor on finitely generated A–modules. Then $\operatorname{Ass}_A F(M_n)$ is ultimately constant.

Main Results

Proof.

Let $f : A \to R$ be the inclusion ring homomorphism. We know that f^*F defined by $f^*F(M) = \varinjlim F(X)$, where X runs through the finitely generated submodule of M, is a coherent functor and hence commutes with direct sums. Now, let

$$\mathrm{Hom}(P, -) \to \mathrm{Hom}(N, -) \to F \to 0$$

be the presentation of F as a coherent functor. Then

$$\oplus_{n\geq 0} \mathrm{Hom}(P, M_n) \to \oplus_{n\geq 0} \mathrm{Hom}(N, M_n) \to f^*F(M) \to 0$$

is exact and since $\mathrm{Hom}(N, M) \cong \oplus_{n\geq 0} \mathrm{Hom}(N, M_n)$ is a finitely generated, it follows that $f^*F(M) \cong \oplus_{n\geq 0} F(M_n)$ is a finitely generated graded R–module with $f^*F(M)_n \cong F(M_n)$. Hence, (by the result of Melkersson 1990) $\mathrm{Ass}_A f^*F(M)_n \cong \mathrm{Ass}_A F(M_n)$ stabilizes for large enough n. □

References

1. M. Brodmann,*Asymptotic stability of Ass*(M/I^nM). Proc. Amer. Math. Soc. 74(1979), 16-18..
2. L. Melkersson, P. Schenzel *Asymptotic prime ideals related derived functors*. Proc. Amer. Math. Soc. 4(1993), 935-938.
3. L. Melkersson, *On asymptotic stability for sets of prime ideals connected with powers of an ideal*. Math. Proc. Camb. Phil. Soc. 107(1990), 267-271.

Impact and Applications of My Research

- The importance of this research is to add new knowledge to the theory of associated primes of finitely generated modules over Noetherian rings.
- It will also equip me with knowledge that will enable me help in training pure mathematicians at my University and Africa at large.

Tack så mycket!

Thank you!