On One Dimensional Dynamical Systems and Commuting Elements in Non Commutative Algebras

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Introduction

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In mathematics, Matrices or more general linear and nonlinear operators do not commute.

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$$XX^* = F(X^*X), \tag{1}$$

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the problem of finding the orbits of F and irreducible representations of (1) are equivalent for important classes of dynamical systems.
We give an explicit description of the interplay between periodic orbits of one-dimensional dynamical systems and commutativity of monomials of linear operators A and B on a Hilbert space $\mathcal{H}$, satisfying the relation

$$AB = BF(A). \tag{2}$$

Definitions

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For two elements $A, B \in \mathcal{A}$, an associative algebra $\mathcal{A}$, we define the commutator $[A : B]$ as $[A : B] = AB - BA$.

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For two elements $A, B \in \mathcal{A}$, an associative algebra $\mathcal{A}$, we define the commutator $[A : B]$ as $[A : B] = AB - BA$.

Then A and B commute if and only if $[A : B] = 0$

Monomial commutativity condition on F

Lemma

Let A and B be bounded operators on a Hilbert space $\mathcal{H}$ satisfying the relation

$$AB = BF(A)$$

for some polynomial F. Then

$$A^j B = BF(A)^j \text{ for all } j \in \mathbb{N} \tag{3}$$

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For all polynomials $\Pi(\lambda) = \sum_{k=0}^{N} \alpha_k \lambda^k$, it holds that

$$\Pi(A)B = B\Pi(F(A)) \tag{4}$$

and consequently

$$A^j B^k = B^k F^{\circ(k)}(A)^j \text{ for all } j, k \in \mathbb{N} \tag{5}$$

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Lemma

Let A and B be bounded operators on a Hilbert space $\mathcal{H}$ satisfying the relation

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for some polynomial F. The monomials $\Phi = B^s A^t$ and $G = B^u A^v$ commute if and only if

$$B^{s+u}\left(F^{o(u)}(A)^t A^v - F^{o(s)}(A)^v A^t\right) = 0. \qquad (6)$$

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Under the assumption of linear independence of normally ordered monomials, the monomials $\Phi = B^s A^t$ and $G = B^u A^v$ commute if and only if

$$F^{o(u)}(A)^t A^v = F^{o(s)}(A)^v A^t \qquad (7)$$

Operators on a finite dimensional space

Suppose A and B are $n-$dimensional operators on the space spanned by the basis $\{e_0, e_1, \cdots, e_{n-1}\}$ defined by;

$$\begin{aligned} Ae_j &= \lambda_j e_j \quad j = 0, 1, \cdots, n-1 \\ Be_j &= e_{j+1} \quad j = 0, 1, \cdots, n-2 \\ Be_{n-1} &= e_0 \end{aligned} \tag{8}$$

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Then $AB = BF(A)$.

Monomial commutativity and periodic points of F

Theorem

Let A and B be operators of the form (8), where $F : [0,1] \to [0,1]$ is any piecewise polynomial map on $I = [0,1]$ and $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ is an $n-$periodic orbit of F. Then we have the following;

If $s \equiv u \pmod{n}$ then $[B^s A^t : B^u A^v] = 0$ if and only if either $t = v$ or if $t \neq v$, one of the following conditions is satisfied

- *$\{\lambda_0, \lambda_1, \cdots, \lambda_{s'-1}\}$ is a periodic orbit of period s', where $s \equiv s' \pmod{n}$, $0 < s' \leqslant n-1$ and s' divides n,*

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- *$\{\lambda_0, \lambda_1, \cdots, \lambda_{n-s'-1}\}$ is a periodic orbit of period $n - s'$ where $s \equiv s' \pmod{n}$, $0 < s' \leq n-1$, $n \leq 2s'$ and $n - s'$ divides both n and s'*

Theorem

If $s \not\equiv u \pmod{n}$ *then* $[B^s A^t : B^u A^t] = 0$ *if and only if* $\{\lambda_0, \cdots, \lambda_{k-1}\}$ *is a periodic orbit of period* k *where* k *is the greatest common divisor of* s', u' *and* n *where* $s \equiv s' \pmod{n}$ *and* $u \equiv u' \pmod{n}$

Crossed product algebras

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$$\mathcal{A} \rtimes_\phi \mathbb{Z} := \{f : \mathbb{Z} \to \mathcal{A} : f(n) = 0 \text{ except for a finite number of } n\}. \quad (9)$$

Endow it with the structure of an associative $\mathbb{C}-$algebra by defining scalar multiplication and addition as the usual pointwise operations.

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Endow it with the structure of an associative $\mathbb{C}-$algebra by defining scalar multiplication and addition as the usual pointwise operations. Multiplication is defined by convolution twisted by the automorphism ϕ as follows;

$$(fg)(n) = \sum_{k\in\mathbb{Z}} f(k)\phi^k(g(n-k)), \tag{10}$$

$f, g \in \mathcal{A} \rtimes_\phi \mathbb{Z},\ n \in \mathbb{Z}$.

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Elements $f, g \in \mathcal{A} \rtimes_\phi \mathbb{Z}$ can be written in the form $f = \sum_{n\in\mathbb{Z}} f_n \delta^n$ and $g = \sum_{n\in\mathbb{Z}} g_m \delta^m$ where $f_n = f(n),\ g_m = g(m)$ and $\delta^n = \chi_{\{n\}}$.

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Addition and scalar multiplication are then canonically defined and multiplication is determined by the relation

$$(f_n\delta^n) * (g_m\delta^m) = f_n\phi^n(g_m)\delta^{n+m} \tag{11}$$

where $m, n \in \mathbb{Z}$ and $f_n, g_m \in \mathcal{A}$.

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In a unital normed algebra, we always assume that $||1_{\mathcal{A}}|| = 1$.

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A Banach $*-$algebra is a complex Banach algebra $\mathcal{A}$ with a conjugate linear involution $*$ (called the **adjoint**) which is an anti-isomorphism. That is, for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

- $(a+b)^* = a^* + b^*$
- $(\lambda a)^* = \bar{\lambda} a^*$
- $a^{**} = a$
- $(ab)^* = b^* a^*$

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- $\mathcal{A} = C(X)$
- $\mathcal{A} = B(\mathcal{H})$

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It has been proven that the crossed product $\mathcal{A} \rtimes_{\phi} \mathbb{Z}$ is a C^*−algebra, *called the C^*−crossed product.*

A maximal abelian subalgebra of $\mathcal{A} \rtimes_\phi \mathbb{Z}$

Definition

1. For a dynamical system $\Sigma = (X, \sigma)$, a point $x \in X$ is called *aperiodic* if for every non-zero $n \in \mathbb{Z}, \ \sigma^n(x) \neq x$.

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Theorem (J. Tomiyama)

The following are equivalent.

1. *Σ is topologically free.*
2. *Every non-zero closed ideal I of $C^*(\Sigma)$ is such that $I \cap C(X) \neq \{0\}$.*
3. *$C(X)$ is a maximal abelian C^*-subalgebra of $C^*(\Sigma)$.*

Automorphisms induced by bijections

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$$\tilde{\sigma}(f) = f \circ \sigma^{-1}$$

and consider the crossed product algebra $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$.

Automorphisms induced by bijections

Definition

For any nonzero $n \in \mathbb{Z}$, set

$$\begin{aligned} Sep^n_{\mathcal{A}}(X) &= \{x \in X \mid \exists\ h \in \mathcal{A} \ :\ h(x) \neq \tilde{\sigma}^n(h)(x)\}, \quad (13) \\ Per^n_{\mathcal{A}}(X) &= \{x \in X \mid \forall h \in \mathcal{A} \ :\ h(x) = \tilde{\sigma}^n(h)(x)\}, \\ Sep^n(X) &= \{x \in X \mid x \neq \sigma^n(x)\}, \\ Per^n(X) &= \{x \in X \mid x = \sigma^n(x)\}. \end{aligned}$$

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All these sets are $\mathbb{Z}-$invariant and if $\mathcal{A}$ separates points of X, (e.g if $\mathcal{A} = C(X)$), then $Sep^n_{\mathcal{A}}(X) = Sep^n(X)$ and $Per^n_{\mathcal{A}}(X) = Per^n(X)$.

Commutants in crossed product algebras

Definition

By the commutant $\mathcal{A}'$ of $\mathcal{A}$ in $\mathcal{A} \rtimes_\phi \mathbb{Z}$ we mean

$$\mathcal{A}' := \{f \in \mathcal{A} \rtimes_\phi \mathbb{Z} : fg = gf \text{ for every } g \in \mathcal{A}\}.$$

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Theorem (C. Svensson, S. D. Silvestrov and M. De Jeu)

The unique maximal abelian subalgebra of $\mathcal{A} \times_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}$ (the commutant) is precisely the set of elements

$$\mathcal{A}' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \textit{for all } n \in \mathbb{Z} : \ f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\}$$

Algebra of piecewise constant functions

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$$\mathcal{A} = \{h \in \mathbb{C}^X : \text{ for every } j \in J : \ h(X_j) = \{c_j\} \text{ for some } c_j \in \mathbb{C}\}.$$

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Lemma

Let $\sigma : X \to X$ be a bijection on X. The following are equivalent.

1. *The algebra $\mathcal{A}$ is invariant under σ and σ^{-1}.*
2. *For every $i \in J$ there exists $j \in J$ such that $\sigma(X_i) = X_j$.*

Commutant of $\mathcal{A}_X$

Recall that the commutant

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Take a partition $\mathbb{P} = \cup_{j\in J} X_j$ of a set X and let $\sigma : X \to X$ be a bijection such that the algebra $\mathcal{A}_X$ is invariant under σ. For $k \in \mathbb{Z}_{>0}$, let

$$C_k := \Big\{ x \in X \mid k \text{ is the smallest positive integer such that } x, \sigma^k(x) \in X_j \text{ for some } j \in J \Big\}. \tag{14}$$

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$$Sep^n_{\mathcal{A}}(X) = \left\{ \bigcup_{k \nmid n} C_k \cup C_\infty \right\}, \tag{15}$$

where C_k be given by (14) *and*

$$C_\infty = \{x \in X \mid \nexists\, j \in J \ : \exists k \ : \ x, \sigma^k(x) \in X_j\}.$$

Description of commutant

Theorem

Let $\mathcal{A}_X$ be the algebra of piece-wise constant functions $f : X \to \mathbb{C}$, $\sigma : X \to X$ any bijection on X such that $\mathcal{A}_X$ is invariant under σ, $\tilde{\sigma} : \mathcal{A}_X \to \mathcal{A}_X$ the automorphism on $\mathcal{A}_X$ induced by σ. Then the unique maximal commutative subalgebra of $\mathcal{A}_X \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}_X$ is given by,

$$\mathcal{A}' = \left\{ \sum_{n\in\mathbb{Z}\ :\ k|n} \left(\sum_{j_n \in J} a_{j_n} \chi_{X_{j_n}} \right) \delta^n \ :\ \text{for all } X_{j_n} \subset C_k \right\}.$$

Applications

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2. Operators satisfying commutation relations of the form

$$XX^* = F(X^*X) \tag{16}$$

or more generally

$$AB = F(BA)$$

play an important role in many models of Physics and Engineering.

Example

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satisfied by operators of creation and annihilation for a system with one degree of freedom. Mathematically, (17) is satisfied if $A(f)(x) = f'(x)$ and $B(f)(x) = xf(x)$ acting on the vector space of infinitely differentiable functions making it important in Physics and many other areas where integration and differentiation are involved.

Tack så mycket!

Thank you!