Global Nonlinear Pricing in the Simplex Method

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Outline

1 Introduction

2 Pricing in the Simplex Method

3 Impact of the Research

4 References

Introduction

Consider the following linear programming problem

$$\begin{aligned} Min \qquad & z = c^T x \\ s.t \qquad & Ax = b \qquad (1) \\ & x \geq 0 \qquad (2) \end{aligned}$$

where A is an mxn matrix with rank m and the vectors are of compatible dimension.
Let us partition the constraint matrix A into a nonsingular mxm matrix B and an $mx(n-m)$ nonbasic matrix n. Then we have

$$A = [B|N], x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$$

Introduction

$$
\begin{aligned}
Ax = b \Rightarrow [B|N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \Rightarrow Bx_B + Nx_N = b \\
\Rightarrow x_B = B^{-1}b - B^{-1}Nx_N = B^{-1}b - \sum_{j \in J} B^{-1}a_j x_j \\
= \overline{b} - \sum_{j \in J} B^{-1}a_j x_j \qquad (3)
\end{aligned}
$$

where J is the set of indices of the current nonbasic variables, $\overline{b} = B^{-1}b$ and the a_j's are the column of A.
By substituting x_B into the partitioned form of the objective function, $z = c_B^T x_B + c_N^T x_N$, we obtain

Introduction

$$
\begin{aligned}
z &= c_B^T \left(B^{-1}b - B^{-1}Nx_N \right) + c_N^T x_N \\
&= c_B^T B^{-1} b + \left(c_N^T - c_B^T B^{-1} N \right) x_N \\
&= z_0 + \sum_{j \in J} \left(c_j - c_B^T B^{-1} a_j \right) x_j \quad \textit{where} \qquad z_0 = c_B^T B^{-1} b \qquad (4)
\end{aligned}
$$

Using the transformation (3) and (4) the linear programming problem LP can be rewritten as :

$$
\begin{aligned}
\textit{Min} \qquad & z = z_0 + \sum_{j \in J} \left(c_j - c_B^T B^{-1} a_j \right) x_j \\
& s.t \sum_{j \in J} \bar{a}_j x_j \leq B^{-1} b \ \textit{where} \ \ \bar{a}_j = B^{-1} a_j \\
& x_j \geq 0, j \in J
\end{aligned}
$$

Reduced Costs

The values $\left(c_j - c_B^T B^{-1} a_j\right)$ which are denoted by $\overline{c}_j$ are called the reduced cost coefficient.

They are the coefficient of the nonbasic variables in the reduced space.

Theorem 1 (Improvement of basic feasible solution)

Given a non-degenerate basic feasible solution with corresponding objective value z_0, suppose that for some k $\overline{c}_k < 0$. Then

- i *there is a feasible solution with objective value $z < z_0$*
- ii *if the variable x_k can be substituted for some vector in the original basis to yield a new basic feasible solution, this new solution will have $z < z_0$*
- iii *if x_k cannot be substituted to yield a basic feasible solution, then the feasible set is unbounded and the objective function can be made arbitrarily small*

Sketch of the Proof

Proof.

Let $x = \begin{bmatrix} x_B \\ 0 \end{bmatrix}$ be a basic feasible solution and suppose that $\overline{c}_k < 0$.

i Now, fixing $x_j = 0$ for $j \in J - \{k\}$,we get

$$z - z_0 = \overline{c}_k x_k \qquad \Rightarrow z < z_0 \qquad using(4)$$

ii From(3) we have,

$$\begin{aligned} & x_{B\,i} = \overline{b}_i - \overline{a}_{ik} x_k & & i = 1, 2, ..., m \\ & If \quad \overline{a}_{ik} > 0, then \;\; x_{B\,i} \geq 0 \\ & \Rightarrow \overline{b}_i - \overline{a}_{ik} x_k \geq 0 \\ & \Rightarrow x_k \leq \frac{\overline{b}_i}{\overline{a}_{ik}} & & i = 1, 2, ..., m \end{aligned}$$

Proof Cont'd

Proof.

$$x_k = \min_{1 \le i \le m} \left\{ \frac{\overline{b}_i}{\overline{a}_{ik}}, \overline{a}_{ik} > 0 \right\} = \frac{\overline{b}_r}{y_{rk}} \text{ for some } r$$

In the absence of degeneracy, $\overline{b}_r > 0$ and hence $x_k > 0$.
Thus the objective function strictly improves as x_k increases from level 0 to $\frac{\overline{b}_r}{y_{rk}}$

iii If $\overline{a}_{ik} \le 0$, then x_{B_i} increases as x_k increases, and so x_{B_i} continues to be non-negative. Thus the problem is unbounded.

□

Optimality Condition

Theorem 2

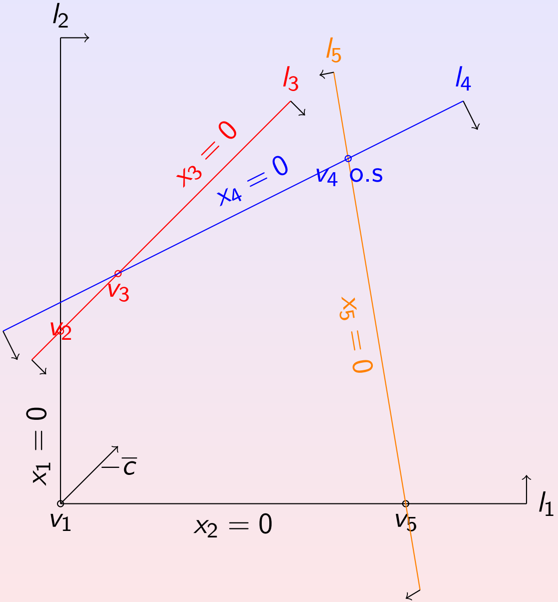
If $\overline{c}_j \geq 0$ for all $j \in J$, then the current basic feasible solution is optimal.

Proof.

Follows from (4) □

Remark

1. If B is not an optimal basis an improving non-basis is selected to enter the basis for the next iteration. This operation is called pricing.
2. All non- basic variables are candidate to enter into the basis. This freedom of choice has led to several pricing strategies.


l_2
l_5
l_3
l_4
$x_3 = 0$
$x_4 = 0$
v_4 o.s
v_3
v_2
$x_5 = 0$
$x_1 = 0$
$-\overline{c}$
v_1
$x_2 = 0$
v_5
l_1

Geometric Motivation of Pricing

Suppose the extreme points or vertices of the feasible region are labeled as $v_1, v_2, .., v_5$ and $\overline{c}$ is the reduced cost.

- v_1 is defined by the hyperplane $x_1 = x_2 = 0$
  $\Rightarrow x_1, x_2 \in N$ and $x_3, x_4, x_5, x_6 \in B$
- v_2, v_3, v_4 and v_5 can be represented in a similar way.

Consider the origin v_1 and the following cases.
Case 1: Holding $x_1 = 0$ and increasing x_2.

- This takes us along the x_2 axis, with the objective function changing at the rate of
$$\frac{\partial z}{\partial x_2} = c_j - c_B^T B^{-1} a_2 = \overline{c}_2 < 0$$
- Since $\frac{\partial z}{\partial x_2} = \overline{c}_2 < 0$, moving along this edge decreases the value of the objective function z.
- Our motion along this edge is blocked by the hyperplane $x_3 = 0$ (moving any further would drive x_3 negative).

- Now, we are at another extreme point of the feasible region v_2 (x_1 and x_3 are nonbasic variables, and x_2, x_4, x_5 and x_6 are basic)
- We call x_2 an entering variable and x_3 a blocking variable or leaving variable.

Case 2: Holding $x_2 = 0$ and increasing x_1.

- This takes us along the x_1 axis, with the objective function changing at the rate of
$$\frac{\partial z}{\partial x_1} = c_1 - c_B^T B^{-1} a_1 = \overline{c}_1 < 0$$
- Again since $\frac{\partial z}{\partial x_1} = \overline{c}_1 < 0$, moving along x_1 decreases the value of the objective function z.
- x_1 is an entering variable and x_5 is a blocking (leaving) variable.

Both cases describes an attractive direction of motion.
This freedom leads us to different pricing strategy.

Pricing Strategies

Some of the known pricing strategies

1. Dantzig Pricing
2. Partial Pricing
3. Sectional Pricing
4. Normalized Pricing

- The best pricing strategy is the one that leads to an optimal solution with least number of iterations.
- Currently none of them is known to be the best one. But one can be superior than the other in a specific problem.
- The selection are good only locally.

Nonlinear Pricing

Consider the following linear programming called P

$$\begin{aligned} Min \qquad & z = c^T x \\ s.t \qquad & Ax - b \leq 0 \\ & x \in X \subseteq \mathbf{R}^n \end{aligned}$$

where A is an mxn matrix with rank m, X is a polytope and the vectors are of compatible dimension.
Let $x_0, x_1, ..., x_{k-1}$ be k grid points of X. Their convex hull is

$$\hat{X} = \{x : x = \sum_{j=0}^{k-1} \lambda_j x_j, \lambda_j \geq 0, \sum_{j=0}^{k-1} \lambda_j = 1\}$$

Global Pricing

Using these points we can formulate the following linear programming called *LPI*, which approximate *P*.

$$
\begin{aligned}
Min \qquad & z = \sum_{j=0}^{k-1} \left(c^T x_j\right) \lambda_j \\
s.t \qquad & \sum_{j=0}^{k-1} (Ax_j)\, \lambda_j - b \leq 0, \\
& \sum_{j=0}^{k-1} = 1 \\
& \lambda_j \geq 0, \qquad\qquad j = 0, 1, ..., k-1
\end{aligned}
$$

Global Pricing

The next grid point x_k is generated by minimizing the following nonlinear program called NLP.

$$Min \qquad L(x,\pi,r) = c^T x + \sum_{i=1}^{m} \left(\frac{\pi_i}{r_i}\right) \left[e^{r_i(A_i x)} - 1\right]$$

$$s.t \qquad x \in X$$

where A_i is the i^{th} row of A, $\pi_i \geq 0$ is the optimal multiplier (optimal dual variable) associated with the i^{th} constraint of (LPI) and $r_i > 0$ are exponential penalty parameter.
The function $L(x,\pi,r)$ is called modified Lagrangian function and solving (NLP) is a global nonlinear pricing operation.

Impact of the Research

Since a large proportion of the total solution time in the simplex method is spent in pricing modifying this step of simplex method has a profound importance both theoretically and practically.

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Tack så mycket!

Thank you!