Decomposition Factors of Perverse Sheaves

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Second Network Meeting for Sida- and ISP-funded PhD Students in Mathematics

Stockholm 26–27 February 2018

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Research Topic

The origin of the theory of perverse sheaves is Goresky and MacPherson's theory of intersection homology.
The aim was to **find a topological invariant similar to cohomology that would carry over some of the nice properties of homology or cohomology of smooth manifolds also to singular spaces** (especially Poincaré Duality).
The intersection homology turns out to be the cohomology of a certain complex of sheaves, constructed by Deligne. This complex is the main example of a perverse sheaf.

Introduction

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A locally constant sheaf on $\mathbb{C}^2 - S$ is given by a complex representation of $\pi_1(\mathbb{C}^2 - S)$, a group that is generated by loops around L_i.
Let $a_i \in \mathbb{C}$ represent the result of the action of a loop around the line L_i. Then, to each multi-index $a = (a_1, ..., a_n) \in \mathbb{C}^n$, there is associated a rank 1 sheaf $\mathcal{L}_a$.

Let j be the inclusion $j : \mathbb{C}^2 - S \hookrightarrow \mathbb{C}^2$, j_* the direct image functor

$$j_* : Sh(\mathbb{C}^2 - S) \to Sh(\mathbb{C}^2)$$

and $R^i j_*$ the i-th right derived functor of j_*

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- **The conditions for its irreducibility;**
- **The number of decomposition factors in the case the previous conditions are not satisfied;**
- **The characterization and description of the support of its decomposition factors.**

Irreducibility

Theorem

The perverse sheaf $Rj_*\mathcal{L}_a$, where $j : \mathbb{C}^2 - S \to \mathbb{C}^2$, is irreducible if, and only if, both of the following conditions are satisfied:

- $a_i \neq 1$, *for all* $i = 1, \ldots, n$;
- $\prod_{i=1}^{n} a_i \neq 1$.

Decomposition Factors

In the case the irreducibility conditions for $Rj_*\mathcal{L}_a$ are not satisfied, we can still ask for the number of decomposition factors of this object. Let $c(P^\bullet)$ represent the number of decomposition factors of the perverse sheaf $P^\bullet$.

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Theorem

Assume that $a_1, \ldots, a_k = 1$ *and* $a_{k+1}, \ldots, a_n \neq 1$.

- *If* $\prod_{i=1}^n a_i = 1$, *then* $c(Rj_*\mathcal{L}_a) = n + k - 1$.
- *If* $\prod_{i=1}^n a_i \neq 1$, *then* $c(Rj_*\mathcal{L}_a) = k + 1$.

Length and decomposition of the cohomology of the complement to a hyperplane arrangement

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Definition

The *Poincaré polynomial* of $\mathcal{A}$ is defined by

$$\Pi(\mathcal{A}, t) = \sum_{X \in L(A)} \mu(X)(-t)^{codim\ X}$$

where μ represents the Möbius function and t is an indeterminate.

Let $U := \mathbb{C}^n \setminus \cup_{i=0}^{m} H_i$ and let $\mathbb{C}_U$ be the constant sheaf on U. We show that the number of decomposition factors of $Rj_*\mathbb{C}_U$ is given by the Poincaré polynomial of the arrangement $\mathcal{A}$.

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Theorem

Let $\mathcal{A}$ be a hyperplane arrangement with hyperplanes H_i, $i = 0, \ldots, m$. Let $j : \mathbb{C}^n \setminus \bigcup_{i=0}^m H_i \to \mathbb{C}^n$ be the inclusion of the complement to the arrangement. Then

$$c(Rj_*\mathbb{C}_U[n]) = \Pi(\mathcal{A}, 1) = \sum_{X \in L(\mathcal{A})} \mid \mu(X) \mid .$$

Let F be a flat (intersection of hyperplanes) and $i : F \to \mathbb{C}^n$ the inclusion. We associate to F the irreducible perverse sheaf

$$N_F = i_* \mathbb{C}_F[\dim_{\mathbb{C}} F] \in Perv(\mathbb{C}^n).$$

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We will describe the decomposition series of $Rj_*\mathbb{C}_U$, as an element in the Grothendieck group $G(\mathcal{A})$ of $Perv(\mathbb{C}^n)$, where $[Rj_*\mathbb{C}_U]$ represents a symbol in $G(\mathcal{A})$.

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Theorem

$$[Rj_*\mathbb{C}_U[n]] = \sum_{F \in L(\mathcal{A})} |\mu(F)|[N_F],$$

where $\mu(F)$ is the Möbius function on the intersection lattice $L(\mathcal{A})$.

Support of the components of the decomposition series of $Rj^*L_\alpha[n]$

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We denote by

$$J(F) = \{H \in \mathcal{A} : H \supset F\}$$

the set of hyperplanes that contain the flat.

Definition

Set

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Theorem

Let $\mathcal{A}$ be a hyperplane arrangement. Then the decomposition series of $Rj_* L_\alpha[n]$ contains an irreducible perverse sheaf with support on the flat F if, and only if, $\alpha \in T_F$.

Impact and Applications of My Research

Which are the good properties of perverse sheaves that make them so attractive?

- The perverse sheaves on a complex manifold X form an abelian category, i.e., the notions of injections, surjections, kernels, cokernels, exact sequences all make sense and have the usual properties.
- The category of perverse sheaves is Artinian, i. e., every perverse sheaf has a finite decomposition series whose successive quotients are irreducible perverse sheaves.
- Perverse sheaves are constructible (or have constructible cohomology sheaves) meaning that there exists a stratification such that for each stratum S, $\mathcal{H}^i(\mathcal{F}_{|S})$ is a local system.

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We extend it to others areas, like combinatorics and algebraic geometry, providing new ways of computing results but also revealing not immediately obvious properties of objects in the different areas involved.

Tack så mycket!

Obrigada!

Thank you!