Root asymptotics for polynomial sequences associated to measures in the complex plane

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Introduction

- Numerous questions have been asked and answered about the relation between zeros and critical points of polynomials $P_n(x)$.
- Oldest being the Rolle's theorem for real zeros and then Gauss-Lucas for complex coefficients.
- Generalizations of Rolle's and Gauss-Lucas theorems have generally been few.

Introduction ct'd

- This is demonstrated by two major challenging open problems, the Sendov conjecture and Steve Smale's conjecture both of which have been proved and studied extensively for special cases but not solved.
- The project is about studying asymptotic distribution of zeros of sequences of univariate polynomials and the distribution of zeros of their derivatives of higher order.

Introduction ct'd

- Let us first handle the problem of zeros of polynomials.
- Many practical applications more especially theoretical physics end up in eigenvalue problems.
- Eigenvalues are zeros of characteristic polynomials and so for at least finite dimensional case there is interest in finding zeros of polynomials.
- In most cases you have knowledge about the asymptotic behaviour of coefficients of the characteristic polynomial and what you want is the information about the zeros.

Introduction ct'd

- Consider a situation where

$$P_n(x) := \prod_{i=0}^{n-1}(x - \frac{i}{n}) \tag{1}$$

whose roots are uniformly distributed in the unit interval.

- We can use the idea of Graffe-Lobachevskii method of separating the root of maximal modulus to find explicit dependence of the root of $P_n(x)$ of maximal modulus x_{max} on n.
- Note that

$$P_n(x) = x^n + \sum_{r=1}^{n-1} a_r x^r. \tag{2}$$

- Then sum of the $j-$th powers of the roots is given by

$$s_j = \sum_{i=1}^{n} x_i^j. \tag{3}$$

- By using Newton sums, Eqn (3) is related to the equation

$$s_j = -ja_j - \sum_{i=1}^{j-1} a_i s_{j-i}. \tag{4}$$

- If there exists a single root of $P_n(x)$ with maximal modulus, then we must have that

$$\lim_{j\to\infty} \frac{s_j}{x_{max}^j} = 1 \Leftrightarrow |s_j|^{1/j} = x_{max}. \tag{5}$$

[1]

Introduction ct'd

- So, if we knew the general expression for s_j, we could find the exact growth of x_{max}. For example in our case,

$$s_j = \sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^j = \frac{n}{1+j} + \dots$$

- In the limiting case we have that

$$\lim_{j\to\infty} \sqrt[j]{s_j} = 1 = \lim_{n\to\infty} \frac{n-1}{n}.$$

- This illustrates the method.

Research Area

We want to consider a similar situation but increasing the order of differentiation along a sequence of polynomials.

- Take a sequence of polynomials $\{P_n\}$;

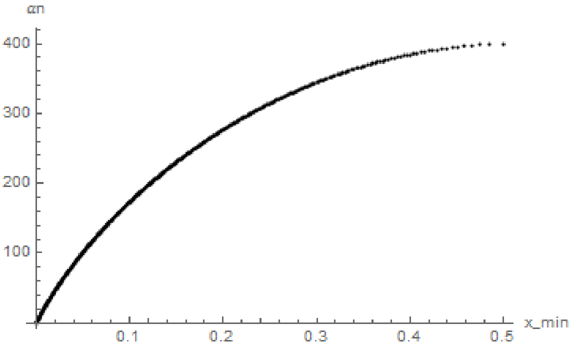
$$P_n(x) := \prod_{i=0}^{n-1} \left(x - \frac{i}{n-1} \right), \quad \deg P_n = n.$$

- For a given positive number $\alpha < 1$, we form the sequence $\{R_n^\alpha\}$ of its derivatives of increasing order, where

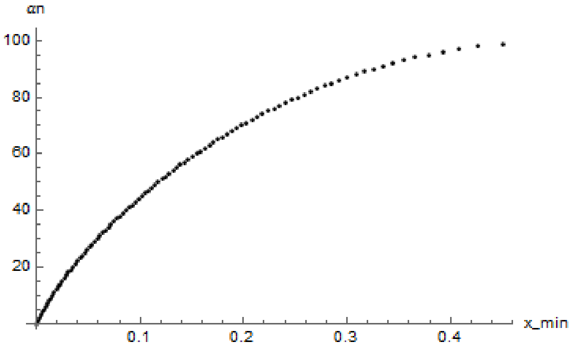
$$R_n^\alpha(x) := P_n^{([\alpha n])}(x).$$

- We want to describe root asymptotics of the sequence $\{R_n^\alpha\}$.

A graph of $[\alpha n]$ against x_{min} for $P_n^{n-[\alpha n]}$ where $n = 400$, $\alpha = \frac{1}{400}, \ldots, \frac{400}{400}$



A graph of $[\alpha n]$ against x_{min} for $P_n^{n-[\alpha n]}$ where $n = 1500$, $\alpha = \frac{1}{100}, \dots, \frac{100}{100}$



Example

- An example of this situation is related to Legendre polynomials and Rodrigues' formula in which case the original probability measure consists of two masses equal to $\frac{1}{2}$ and supported at ± 1.
- Then $P_{2n} = (z^2 - 1)^n$ and for the sequence of Legendre polynomials
$$\mathfrak{P}_n := \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left((z^2 - 1)^n \right).$$
[2]
- The density of their asymptotic root distribution equals
$$\rho(x) = \frac{2}{\pi} \sqrt{1 - x^2} dx, \ x \in [-1, 1].$$

Example

- More generally, for a given $0 < \alpha < 2$, if we consider the sequence

$$R_n^\alpha(z) := \frac{\mathrm{d}^{[\alpha n]}}{\mathrm{d}z^{[\alpha n]}} \left((z^2 - 1)^n\right),$$

- The asymptotic root distribution will be supported on the interval

$$I_\alpha = [-\sqrt{\alpha(2-\alpha)}, \sqrt{\alpha(2-\alpha)}]$$

with the density given by

$$\rho^\alpha(x) = \frac{2}{\pi}\sqrt{2\alpha - \alpha^2 - x^2}, \quad x \in I_\alpha.$$

- In particular, for all $0 < \alpha < 2$, I_α is contained in $[-1, 1]$ and coincides with it only for $\alpha = 1$.

References

- B. Shapiro, and M. Tater, Asymptotics of spectral polynomials, Acta Polytechnica vol 47(2-3) (2007) 32-35.
- R. Bøgvad, Ch. Hägg, and B. Shapiro, Around Rodrigues' formula.

Thank you!