Orthogonal Polynomials, Operators and Commutation Relations

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Sida

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Research Topic – Analysis/Algebra

Orthogonal polynomials, operators and commutation relations appear in many areas of mathematics, physics and engineering where they play a vital role.

Research Topic – Analysis/Algebra

Orthogonal polynomials, operators and commutation relations appear in many areas of mathematics, physics and engineering where they play a vital role.

- Orthogonal functions $\rightarrow$ Fourier series and wavelets $\rightarrow$ signal processing.
- Orthogonal polynomials $\rightarrow$ L^2-boundedness of singular integral operators.
- L^p-convergence of Fourier series $\Leftrightarrow$ L^p-boundedness of singular integrals.
- Position and momentum operators $[x, p] \equiv xp - px = ih$

$$[x_i, p_j] = ih\delta_{ij}, \tag{1}$$
$$[x_i, x_j] = [p_i, p_j] = 0. \tag{2}$$

Creation and annihilation operators; Bosons, replace $ih\delta_{ij}$ with δ_{ij} in (1).
Fermions, replace $[\ ,\ ]$ with $\{\ ,\ \}$.

Background

Sten Kaijser (1999), *Några nya ortogonala polynom*, Normat

- $\{\tau_n\}$ – orthogonal on the real line $\mathbb{R}$ with respect to

$$\omega_1(x) = \frac{1}{2\cosh\frac{\pi}{2}x}. \tag{3}$$

- $\{\sigma_n\}$ – orthogonal in the strip

$$\mathbb{S} = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\} \tag{4}$$

with respect to ω_1.

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- Connected by two operators,

$$Rf(x) = \frac{f(x+i) + f(x-i)}{2}, \tag{5}$$

$$Jf(x) = \frac{f(x+i) - f(x-i)}{2i}. \tag{6}$$

Motivation

- $$\tau_n \xrightarrow{\quad R \quad} \rho_n \tag{7}$$
  $$\sigma_{n+1} \xleftarrow{\quad Q \quad} \rho_n \tag{8}$$
  where $Qf(x) = xf(x)$.

- By Favard's condition, Araaya $\to$ ρ-polynomials were orthogonal;

  But the weight was not known.

  Lars Holst (2013), *Probabilistic proofs of Euler identities*, J. Appl. Probab. 50 no. 4, 1206–1212.

- $\{\rho_n\}$ – orthogonal on the real line $\mathbb{R}$ with respect to

$$\omega_2(x) = (\omega_1 * \omega_1)(x) = \frac{x}{2\sinh\frac{\pi}{2}x} \tag{9}$$

Connections between the systems

Three Meixner-Pollaczek polynomials connected by three operators

σ	τ	ρ
$\sigma_0 = 1$	$\tau_0 = 1$	$\rho_0 = 1$
$\sigma_1 = x$	$\tau_1 = x$	$\rho_1 = x$
$\sigma_2 = x^2$	$\tau_2 = x^2 - 1$	$\rho_2 = x^2 - 2$
$\sigma_3 = x^3 - 2x$	$\tau_3 = x^3 - 5x$	$\rho_3 = x^3 - 8x$
$\sigma_4 = x^4 - 8x^2$	$\tau_4 = x^4 - 14x^2 + 9$	$\rho_4 = x^4 - 20x^2 + 24$
$\vdots$	$\vdots$	$\vdots$

$$Rf(x) = \frac{f(x+i) + f(x-i)}{2} \qquad \sigma_n \xrightarrow{R} \tau_n \xrightarrow{R} \rho_n$$

$$Jf(x) = \frac{f(x+i) - f(x-i)}{2i} \qquad \sigma_n \xrightarrow{J} n\tau_{n-1} \xrightarrow{J} n(n-1)\rho_{n-2}$$

$$Qf(x) = xf(x) \qquad \sigma_{n+1} \xleftarrow{Q} \rho_n$$

Paper I

1. Introduces the ρ-system and establishes its connection to the Sten–Araaya systems in terms of the operators R, J and Q.

2. Investigates boundedness properties of two other operators, $B = R^{-1}$ and $S = JR^{-1}$, both as convolution operators,

$$Bf(z) = \int_{-\infty}^{\infty} \frac{f(t)dt}{2\cosh\frac{\pi}{2}(z-t)} \quad \text{and} \quad Sf(x) = \lim_{\varepsilon\to 0^+} \int_{|x-t|>\varepsilon} \frac{f(t)dt}{2\sinh\frac{\pi}{2}(x-t)},$$

in the Hilbert spaces related to the three systems.

— Achieved: Proved that both operators are bounded on L^2-spaces, and estimates of the norms are obtained.

— How?

Weighted spaces – Orthogonal polynomials

Unweighted spaces – Fourier analysis

Paper II

- Extends the investigation of the boundedness properties of the operators B and S to L^p-spaces ($1 < p < \infty$).

- Achieved: Proved that both operators are bounded on $L^p(\mathbb{R})$ and $L^p(\omega_1)$, and estimates of the norms are obtained.

- How?: By proving boundedness for $p = 2$ and weak boundedness for $p = 1$, and then using interpolation to obtain boundedness for $1 < p \leq 2$.

- What of for $2 \leq p < \infty$?:

  Unweighted case – Duality

  Weighted case – Methods of M. Riesz (1928) for the conjugate function operator.

Paper III

- Determination of all nonnegative functions ω for which there is a constant C such that

$$\int_{-\infty}^{\infty} [Sf(x)]^p \omega(x)\, dx \leq C \int_{-\infty}^{\infty} \left|f(x)\right|^p \omega(x)\, dx, \qquad (10)$$

  where $1 < p < \infty$, C is independent of f and S is the operator defined by

$$Sf(x) = \lim_{\varepsilon \to 0^+} \int_{|x-t|>\varepsilon} \frac{f(t)dt}{2\sinh \frac{\pi}{2}(x-t)}.$$

- The main result is that ω is such a function if and only if

$$\left(\int_I \omega(x)\, dx\right)\left(\int_I [\omega(x)]^{-1/(p-1)}\, dx\right)^{p-1} \leq K|I|^p, \qquad (11)$$

  where I is any interval and K is a constant independent of I.

- Related problems also considered: Weak type results, the case when $p = 1$ or $p = \infty$, and the result when S is replaced by B.

Paper IV

- Reordering formulas – Application: Centralizers and centers
- Motivation – The operators J, R and Q satisfy

$$QJ - JQ = -R, \tag{12}$$

$$QR - RQ = J, \tag{13}$$

$$JR - RJ = 0. \tag{14}$$

- Normal order: JRQ. Normalised: $\sum_{k=0}^{n} p_k(J,R)Q^k$
- For all polynomials $p(J,R)$,

$$\left[Q, p(J,R)\right] = -R\frac{\partial p(J,R)}{\partial J} + J\frac{\partial p(J,R)}{\partial R}. \tag{15}$$

- For all nonnegative integers k, m and n,

$$Q^k S^m T^n = S^m T^n \left(Q + (m-n)i\right)^k, \tag{16}$$

where $S = J + iR$ and $T = J - iR$.

Paper V

For some bijective function $\sigma : X \to X$, $X \subset \mathbb{C}$, define R, J, Q by

$$Rf(x) = \frac{f(\sigma(x)) + f(\sigma^{-1}(x))}{2}, \tag{17}$$

$$Jf(x) = \frac{f(\sigma(x)) - f(\sigma^{-1}(x))}{2i}, \tag{18}$$

$$Qf(x) = xf(x) \quad \Longrightarrow \sigma(Q)f(x) = \sigma(x)f(x). \tag{19}$$

- We then have

$$JR = RJ, \tag{20}$$

$$RQ = \frac{\sigma(Q) + \sigma^{-1}(Q)}{2} R - \frac{\sigma(Q) - \sigma^{-1}(Q)}{2i} J, \tag{21}$$

$$JQ = \frac{\sigma(Q) + \sigma^{-1}(Q)}{2} J + \frac{\sigma(Q) - \sigma^{-1}(Q)}{2i} R. \tag{22}$$

- Reordering formulas – Application: Centralizers and centers

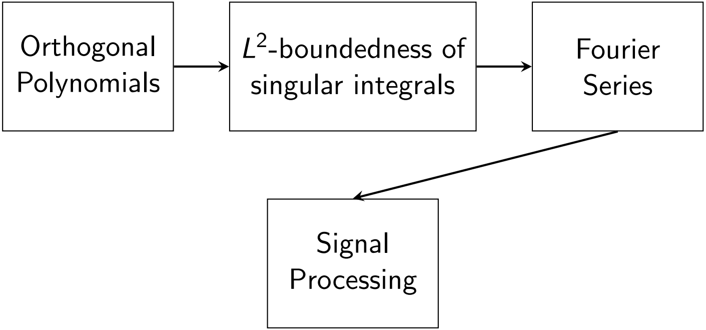
Impact and Applications of My Research

- Contribution to research in mathematics at my university

- Notions of harmonic analysis, noncommuatative analysis and deformations of algebras to my univesrity and country.

- Bridges the gap between pure and applied mathematics.

  For instance, the research demonstrates that:



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Tack så mycket!

Thank you!