Decomposition Methods For Solving Large-Scale Optimization Problems

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Introduction

- Large complex combinatorial optimization problems.
- Difficult to solve (exact methods not feasible).
- Develop heuristics:
  - Finds near-optimal solutions.
  - Based on mathematical optimization models and Lagrangian theory.

General Convex Problem

- Primal convex problem

$$f^* = \min \quad f(x)$$
$$\text{s.t.} \quad g(x) \leq 0 \tag{1}$$
$$x \in X$$

where $g = (g_1, g_2, .., g_m)$

- f and $g_i, i = 1, 2, ..., m$, are convex functions, X is a convex set.
- Given $u \geq 0$, the Lagrangian dual function is

$$h(u) = \min_{x \in X} \{f(x) + u^T g(x)\} \tag{2}$$

- The Lagrangian dual problem

$$h^* = \max_{u \geq 0} h(u) \tag{3}$$

Global Optimality Conditions

- Let u^* be optimal solution in (3)
- An $x \in X$ is optimal iff it satisfies the conditions

$$f(x) + {u^*}^T g(x) \leq h(u^*) \tag{4a}$$

$$g(x) \leq 0 \tag{4b}$$

$$ {u^*}^T g(x) = 0 \tag{4c}$$

Example (Non-Convex Problem)

- Example

$$\begin{aligned} f^* = \min \quad & f(x) = -x_2 \\ \text{s.t.} \quad & g(x) = x_1 + 4x_2 - 6 \leq 0 \\ & x \in X = \{x \in \mathbb{Z}^2 | 0 \leq x_1 \leq 4, 0 \leq x_2 \leq 2\} \end{aligned} \tag{5}$$

- Given $u \geq 0$, the Lagrangian dual function is

$$h(u) = -6u + \min_{x \in X}\{ux_1 + (4u-1)x_2\} \tag{6}$$

$$\Rightarrow h(u) = \begin{cases} 2u - 2 & \textit{for} \quad 0 \leq u \leq \frac{1}{4} \\ -6u & \textit{for} \quad \frac{1}{4} \leq u \end{cases}$$

Example Cont'd

- The maximum value of $h(u)$ is obtained when $u^* = \frac{1}{4} \quad \Rightarrow h(u^*) = -\frac{3}{2}$.
- A primal optimal solution: $x^* = (1, 1)^T$ and $f^* = -1$
- Conditions $(4a)$ and $(4b)$ are satisfied, but not $(4c)$ since $u^T g(x) = -\frac{1}{4} \neq 0$.

Non-Convex Problem

$$\begin{aligned} f^* = \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \\ & x \in X \end{aligned} \tag{7}$$

where $g = (g_1, g_2, .., g_m)$

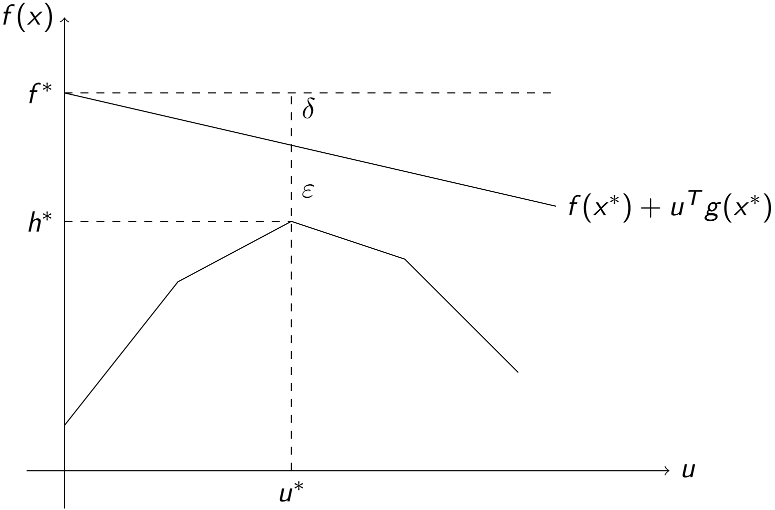
- f and $g_i, i = 1, 2, ..., m$, are convex functions, X a finite set (non-convex).
- Duality gap: $\Gamma = f^* - h^*$.

Generalized Global Optimality Conditions

- Let u^* be dual optimal solution
- Generalized optimality conditions

$$f(x) + {u^*}^T g(x) \leq h(u^*) + \varepsilon \tag{8a}$$
$$g(x) \leq 0 \tag{8b}$$
$${u^*}^T g(x) \geq -\delta \tag{8c}$$
$$\varepsilon + \delta \leq \Gamma \tag{8d}$$
$$\varepsilon, \delta \geq 0 \tag{8e}$$

Non-Convex Problem



Example Revisited

From previous example

- The maximum value of $h(u)$ is obtained when $u^* = \frac{1}{4}$
  $\Rightarrow h(u^*) = -\frac{3}{2}$.
- A primal optimal solution is $x^* = (1, 1)^T$.
- x^* satisfies the conditions $(8a) - (8e)$ with $\varepsilon = \frac{1}{4}$ and $\delta = \frac{1}{4}$
- Note that $\varepsilon + \delta = f^* - h^* = \Gamma = \frac{1}{2}$

Cartesian Product With Coupling Constraints

Consider a general integer programming problem

$$\min \quad \sum_{j=1}^{n} c_j^T x_j \tag{9a}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} A_j x_j \geq b \tag{9b}$$

$$x_j \in X_j, \quad j = 1, 2, ..., n \tag{9c}$$

where $c_j \in R^{n_j}$, $A_j \in R^{m \times n_j}$, $X_j \subseteq R^{n_j}$ is a finite set and $b \in R^m$.

Lagrangian Relaxation

- Given $u \geq 0$, the dual function is

$$
\begin{aligned}
h(u) = \min \quad & \sum_{j=1}^{n} c_j^T x_j + u^T (b - \sum_{j=1}^{n} A_j x_j) \\
s.t. \quad & x_j \in X_j, \quad j = i, ..., n.
\end{aligned}
$$

$$
= b^T u + \sum_{j=1}^{n} \underbrace{\min_{x_j \in X_j} (c_j^T - u^T A_j) x_j}_{h_j(u)}
$$

- The Lagrangian dual problem

$$
h^* = \max_{u \geq 0} \quad h(u) = b^T u + \sum_{j=1}^{n} h_j(u) \qquad (10)
$$

Global Optimality conditions

- Let u^* be optimal solution in (10)
- The global optimality conditions are

$$(c_j^T - {u^*}^T A_j)x_j \leq h_j(u^*) + \varepsilon_j, \quad \forall j \tag{11a}$$

$$b - \sum_{j=1}^{n} A_j x_j \leq 0 \tag{11b}$$

$$u_i^*(b - \sum_{j=1}^{n} A_j x_j)_i \geq -\delta_i, \quad \forall i \tag{11c}$$

$$\sum_{j=1}^{n} \varepsilon_j + \sum_{i=1}^{m} \delta_i \leq \Gamma, \quad \forall i, j \tag{11d}$$

$$\varepsilon_j, \delta_i \geq 0, \quad \forall i, j \tag{11e}$$

Motivation for Heuristic Method

- Given $u \geq 0$ and $x_j \in X_j, \forall j$, define

$$\varepsilon_j(x_j, u) = (c_j^T - u^T A_j)x_j - h_j(u) \tag{12}$$

and

$$\delta_i(x, u_i) = -u_i(b - \sum_{j=1}^{n} A_j x_j)_i \tag{13}$$

Motivation for Heuristic Method Contn'd

- $\varepsilon_j(x_j, u)$ is the degree of near-optimality of a feasible solution x in the Lagrangian dual problem.
- $\delta_i(x, u_i)$ is the degree of near-complementarity of the feasible solution x.
- To get a near optimal primal solution, the following need to be fulfilled:
  - Near-optimality in the Lagrangian dual problem.
  - Near-complementarity.
  - Primal feasibility.
- We want all $\varepsilon_j(x_j, u)$ and all $\delta_i(x, u_i)$ to be small
- Use heuristics to achieve feasibility.
- We develop a new problem which minimizes a weighted objective function of all $\varepsilon_j(x_j, u)$ and $\delta_i(x, u_i)$.

Auxiliary Problem

- Given $\alpha_j \geq 0, \forall j$ and $\beta_i \geq 0, \forall i$ and a $\overline{u} \geq 0$. Then the auxiliary problem is as follows

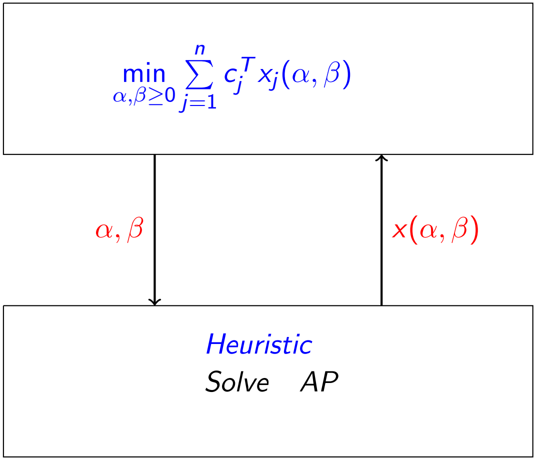
$$(AP) \quad P(x, \overline{u}, \alpha, \beta) = \min \quad \sum_{j=1}^{n} \alpha_j \varepsilon_j(x_j, \overline{u}) + \sum_{i=1}^{m} \beta_i \delta_i(x, \overline{u}_i) \tag{14a}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} A_j x_j \geq b, \tag{14b}$$

$$x_j \in X_j, \quad \forall j \tag{14c}$$

- Solve the (AP) heuristically.
- Let $x(\alpha, \beta)$ be the solution found.

Solution Methods



Tack så mycket!

Thank you!