Algebraic Combinatorics

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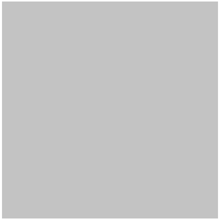

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Research Topic

In my research I will concentrate on Coxeter groups and Combinatorics of Coxeter groups. So far I am reading on finite reflection Groups (finite Coxeter Groups).
I have to do course works equivalent to 120 Credits and to publish some research papers. In order to conduct well my research I will use published research papers and books related to my area of research.
Down I can talk about what I am doing in my research area in the first three months of my studies:

Summary of what I am doing

Definition

A reflection is a linear operator s on a real vector space V which sends a non zero vector α to $-\alpha$, while fixing pointwise the hyperplane H_α orthogonal to α. The reflection of λ is given by the formula

$$s_\alpha \lambda = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha$$

In fact s_α is an orthogonal transformation, i.e., $(s_\alpha \lambda, s_\alpha \mu) = (\lambda, \mu)$ for all $\lambda, \mu \in V$. Also $s_\alpha^2 = 1$, and hence s_α has order 2 in the group $O(V)$ of all orthogonal transformations of V. A finite reflection group is a type of subgroups of $O(V)$.
Down I can list some examples:

Example

- Symmetric group S_n. This is a group of permutations of a set on n numbers $X = \{1, 2, \cdots, n\}$. It is generated by transpositions $(i, i+1)$, where $1 \leq i \leq n-1$.
- Dihedral group D_m of order $2m$. It consists of the orthogonal transformations which preserves a regular m-sided polygon centered at the origin. D_m contains m rotations and m reflections. It is generated by reflections because a rotation through $2\pi/m$ can be achieved as a product of two reflections relative to a pair of adjacent diagonals which meet at an angle of $\theta = \pi/m$.

In order to analyze finite reflection groups one must refer to the root system Φ.

Root system

The root system ϕ is a finite set of nonezero vectors in V satisfying the conditions:

(1) $\phi \cap \mathbf{R} = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;

(2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Then we define finite reflection groups generated by all reflections s_α, $\alpha \in \phi$.

Generators and relations in a Reflection group W

Fixing a simple system Δ in ϕ, the reflection group W is generated by the set $S := \{s_\alpha, \alpha \in \Delta\}$, subjected to the relations:

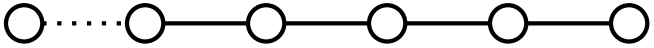
$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, (\alpha, \beta) \in \Delta.$$

Here Δ is is a vector space basis for the $\in$ **R**-span of ϕ in V and each $\alpha \in \phi$ is a linear combination of Δ with coefficients all of the same sign. Δ is called a simple system. $m(\alpha, \beta)$ is the order of the element $s_\alpha s_\beta \in W$. Finite reflection groups (Coxeter groups) are related to some positive definite graphs in the following way:

Positive definite Cxeter graph

A Coxeter graph is a finite (undirected graph) graph, whose edges are labelled with integer $\geqslant 3$ or with the symbol ∞. Let S be the set of generators, and $m(s, s')$ be the label on the edge joining $s \neq s'$. Let also $m(s, s') = 2$ for vertices $s \neq s'$ not joined by an edge. Then the Coxeter graph with vertex set S of cardinality n is associated with the symmetric $n \times n$ matrix A where each entry in it is of the form $a(s, s') = -cos(\frac{\pi}{m(s,s')})$. If $x^t Ay > 0$ where $x, y \in \mathbf{R}^n$ then A is positive definite. A Coxeter graph whose associated matrix is positive definite is a positive definite graph.

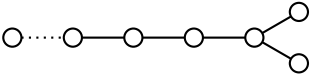
Some positive definite graphs corresponding to some Coxeter groups

$A_n, n \geqslant 1$ 

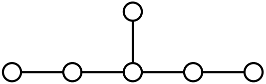
It represents a set of generators of S_{n+1} with $(s_i s_{i+1})^3 = 1$, $(s_i s_i)^2 = 1$ and if $j \geqslant i + 2$ $s_i s_j = s_j s_i$. Here each s_i is a node corresponding to a simple reflection of a Coxeter group S_{n+1}. The label between each two adjacent nodes is 3 (which is the order of $s_i s_{i+1}$). Here the root system Φ consists of $n(n+1)$ roots and S_{n+1} has order $(n+1)!$.

Some positive definite graphs corresponding to some Coxeter group

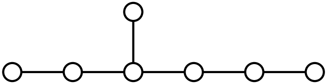
D_n, $n \geqslant 4$



E_6



E_7



$\lvert W \rvert$ and $\lvert \Phi \rvert$ for some Coxeter groups			
A_n	D_n	E_6	E_7
$(n+1)!$	$2^{n-1}n!$	$2^7 3^4 5$	$2^{10} 3^4 5 7$
$n(n+1)$	$2n(n-1)^2$	72	126

Applications of My Research

- Coxeter groups arise as symmetric groups of regular polytopes and as Weyl groups associated to root systems, which in turn are associated to Lie groups, Lie algebras, and/or algebraic groups; In fact the Coxeter groups are very important in understanding the Lie groups, Lie algebras, algebraic groups and combinatorics.
- For example quasicrystals and fullerenes can be modeled using affine three-dimensional Coxeter groups.

Tack så mycket!

Thank you!