Single-Pass Observation Update of Smoothing Posterior

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Problem Formulation

A method is derived for updating the smoothing posterior of a linear Gaussian state-space model in a single pass at the reception of a new observation. Commonly the posterior distribution would be recomputed using a two-pass formula.

Proposition

Given is the distribution $p(\mathcal{X}|\mathcal{Y}) = \mathcal{N}(\mathcal{X}|\hat{\mathcal{X}}, \mathbf{P})$, represented and uniquely determined by

$$\begin{aligned} \hat{\mathbf{x}}_k &= \mathrm{E}\big(\mathbf{x}_k \,|\, \mathcal{Y}\big), && k \in 0 \cup \mathcal{K}, \\ \mathbf{P}_k &= \mathrm{Cov}\big(\mathbf{x}_k \,|\, \mathcal{Y}\big), && k \in 0 \cup \mathcal{K}, \\ \mathbf{P}_{k-1,k} &= \mathrm{Cov}\big(\mathbf{x}_{k-1}, \mathbf{x}_k \,|\, \mathcal{Y}\big), && k \in \mathcal{K}, \end{aligned}$$

A new observation is obtained at time step $\tau \in 0 \cup \mathcal{K}$ as

$$\mathbf{z} = \mathbf{H}\mathbf{x}_\tau + \mathbf{e}, \quad \mathbf{e} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}).$$

The posterior $p(\mathcal{X}|\mathcal{Y}, \mathbf{z}) = \mathcal{N}(\mathcal{X}|\hat{\mathcal{X}}^+, \mathbf{P}^+)$ is then given by

$$\begin{aligned} \hat{\mathbf{x}}_\tau^+ &= \hat{\mathbf{x}}_\tau + \mathbf{P}_\tau \mathbf{H}^T \mathbf{S}_\tau^{-1}(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}_\tau), \\ \mathbf{P}_\tau^+ &= \mathbf{P}_\tau - \mathbf{P}_\tau \mathbf{H}^T \mathbf{S}_\tau^{-1} \mathbf{H} \mathbf{P}_\tau, \end{aligned}$$

for the state with index τ, where $\mathbf{S}_\tau = \mathbf{H}\mathbf{P}_\tau\mathbf{H}^T + \mathbf{R}$, and is recursively computed backwards by

$$\begin{aligned} \hat{\mathbf{x}}_k^+ &= \hat{\mathbf{x}}_k + \mathbf{K}_k^b(\hat{\mathbf{x}}_{k+1}^+ - \hat{\mathbf{x}}_{k+1}), \\ \mathbf{P}_{k,k+1}^+ &= \mathbf{P}_{k,k+1} + \mathbf{K}_k^b(\mathbf{P}_{k+1}^+ - \mathbf{P}_{k+1}), \\ \mathbf{P}_k^+ &= \mathbf{P}_k + (\mathbf{P}_{k,k+1}^+ - \mathbf{P}_{k,k+1})(\mathbf{K}_k^b)^T, \end{aligned}$$

for $k < \tau$, where $\mathbf{K}_k^b = \mathbf{P}_{k,k+1}\mathbf{P}_{k+1}^{-1}$ and forwards by

$$\begin{aligned} \hat{\mathbf{x}}_k^+ &= \hat{\mathbf{x}}_k + \mathbf{K}_k^f(\hat{\mathbf{x}}_{k-1}^+ - \hat{\mathbf{x}}_{k-1}), \\ \mathbf{P}_{k-1,k}^+ &= \mathbf{P}_{k-1,k} + (\mathbf{P}_{k-1}^+ - \mathbf{P}_{k-1})(\mathbf{K}_k^f)^T, \\ \mathbf{P}_k^+ &= \mathbf{P}_k + \mathbf{K}_k^f(\mathbf{P}_{k-1,k}^+ - \mathbf{P}_{k-1,k}). \end{aligned}$$

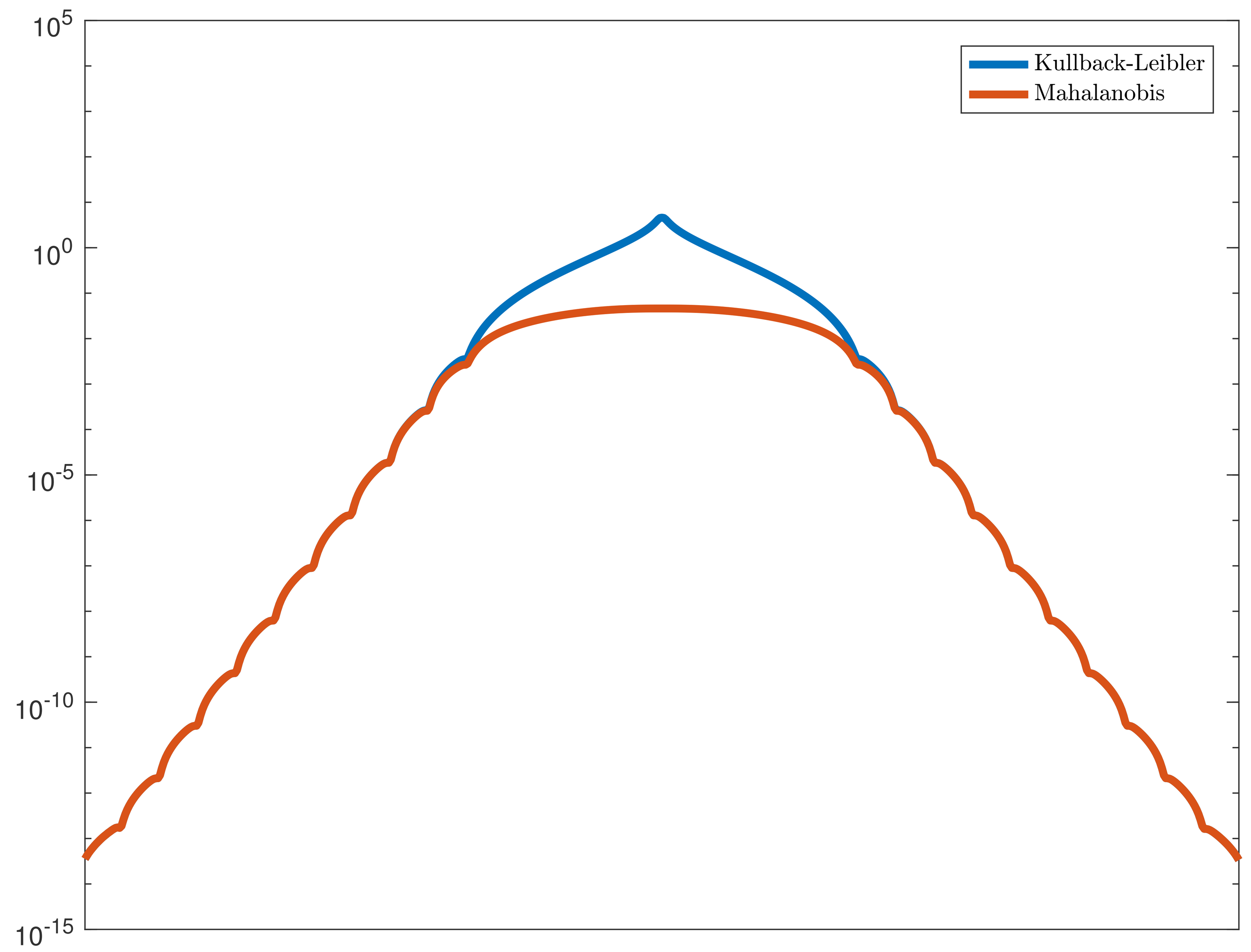
for $k > \tau$, where $\mathbf{K}_k^f = \mathbf{P}_{k-1,k}^T \mathbf{P}_{k-1}^{-1}$.

Early Termination

The Kullback-Leibler divergence and Mahalanobis distance

$$\begin{aligned} D_{KL} &= \frac{1}{2}\Big(D_M - \mathrm{tr}\,(\mathbf{Q}_k) - \log\big(\det\,(I - \mathbf{Q}_k)\big)\Big), \\ D_M &= (\hat{\mathbf{x}}_k^+ - \hat{\mathbf{x}})^T(\mathbf{P}_k^+)^{-1}(\hat{\mathbf{x}}_k^+ - \hat{\mathbf{x}}), \end{aligned}$$

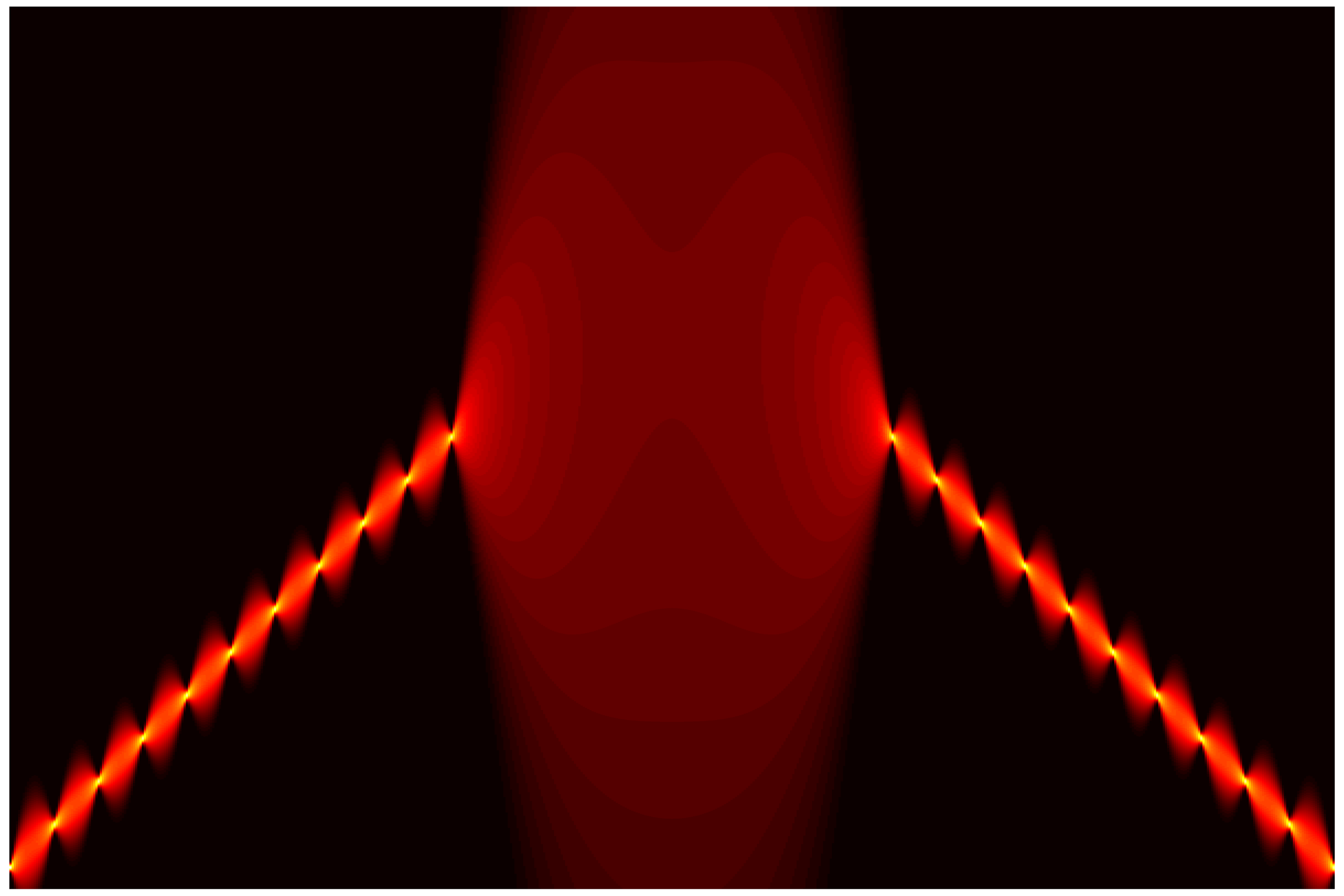
where $\mathbf{Q}_k = (\mathbf{P}_k^+)^{-1}(\mathbf{P}_k^+ - \mathbf{P}_k)$, can be used to terminate the recursions early by comparing to a threshold.



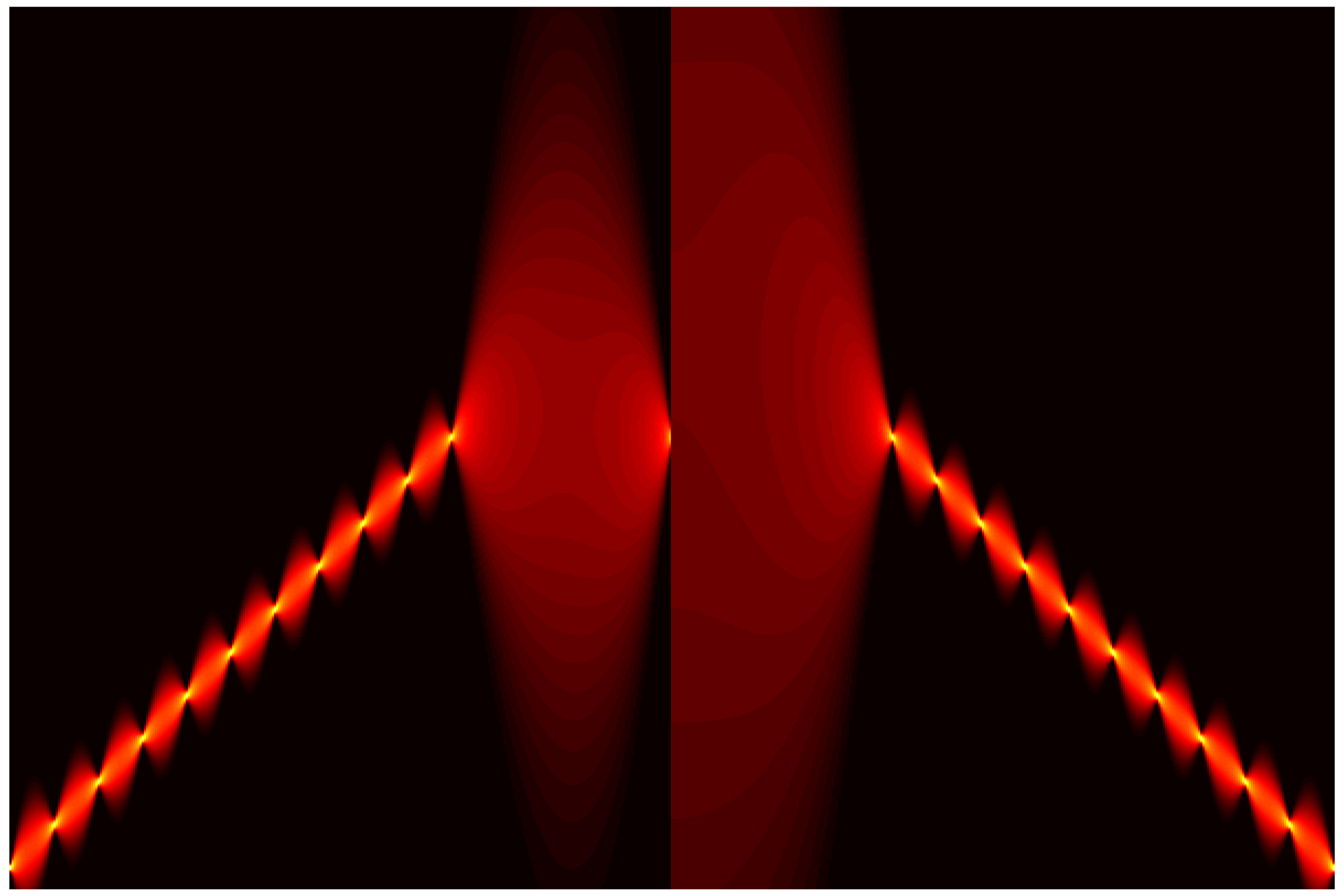
Kullback-Leibler divergence and half Mahalanobis distance.

Summary

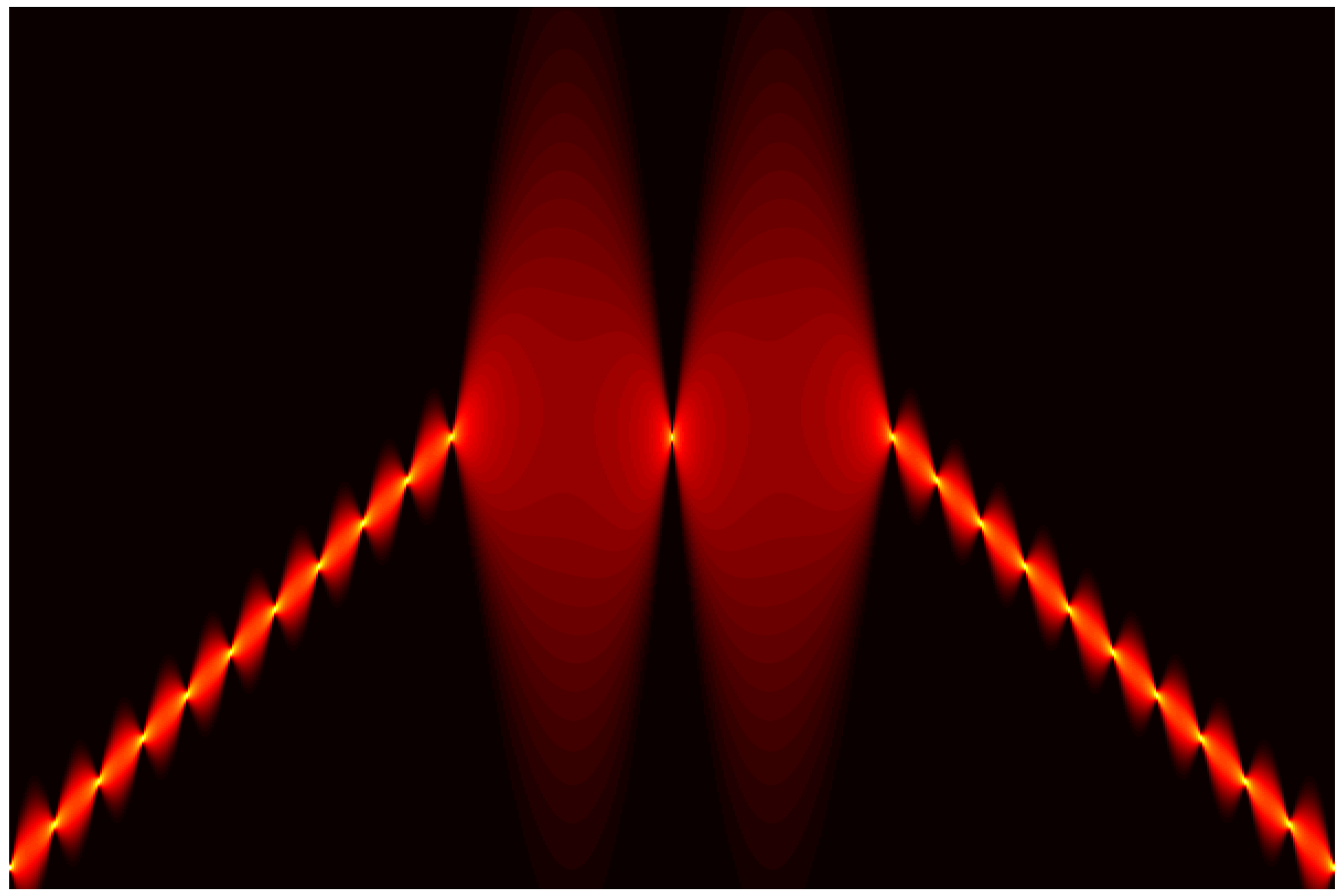
- A single-pass update of smoothing posterior is derived.
- Early algorithm termination is proposed for speed-up.



Original distribution.



Distribution after backward recursion.



Posterior distribution.

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