Title: A Qualitative Description of Extremals for Morrey's Inequality.

Abstract: The seminorm form of Morrey's inequality is summarized as follows: Let $u \in L^1_{loc}(R^n)$ be such that $Du \in L^p(R^n)$ and $p > n$. Then there is some $C > 0$ depending only on n and p such that

$$C\|Du\|_p \geq [u]_{C^{0,1-n/p}} \tag{0.1}$$

where $[\cdot]_{C^{0,1-n/p}}$ is the $C^{0,1-n/p}$-Holder seminorm given by $[u]_{C^{0,1-n/p}} := \sup_{x\neq y}\left\{\frac{|u(x)-u(y)|}{|x-y|^{1-n/p}}\right\}$. This inequality was (essentially) proven 80 year ago by C. B. Morrey Jr. However, until recently, nothing was known about extremals or the sharp constant of Morrey's inequality. In a recent project, R. Hynd and I proved the existence of extremals and some of their qualitative characteristics. The key to our results is to show that a function, v, is an extremal of Morrey's Inequality if and only if it satisfies a PDE:

$$-\Delta_p v = c(\delta_x - \delta_y) \tag{0.2}$$

where δ_x and δ_y are dirac masses at some $x, y \in \mathbb{R}^n$ and c is any nonzero constant. The points x and y in (0.2) are essential in the structure of v. In a recent project, we show that x and y are the unique pair of points where v achieves its $C^{0,1-n/p}$-Holder seminorm, they are the points where v achieves its absolute maximum and minimum, and v is analytic except at x and y. Moreover, using the PDE, (0.2), we are able to show that extremals of Morrey's inequality are cylindrically symmetric (if $n \geq 3$) or evenly symmetric (if $n = 2$) about the line containing x and y, reflectionally antisymmetric (up to addition by a constant), and unique up to operations that are invariant on the ratio of the seminorms in (0.1). We also give explicit solutions for extremals when $n = 1$ and some numeraical approximations of extremals for $n = 2$ and $p = 4$. This work is a collaboration with R. Hynd.