Conservative Linear Unbiased Estimator (CLUE)

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Background

Assume $\mathbf{y} = \mathbf{Hx} + \mathbf{v}$ where $\mathbf{x}$ is the true state and $\mathbf{v}$ is zero-mean random noise. In **classical** linear estimation an estimate $\hat{\mathbf{x}}$ is calculated linearly from $\mathbf{y}$ with $\mathbf{R} = \mathrm{cov}(\mathbf{y})$, *i.e.*,

$$\hat{\mathbf{x}} = \mathbf{Ky}, \qquad \mathbf{P} = \mathrm{cov}(\hat{\mathbf{x}}) = \mathrm{cov}(\mathbf{Ky}) = \mathbf{KRK}^\top.$$

In **conservative** linear estimation it is only known that $\mathbf{R} \in \mathfrak{R}$ where $\mathfrak{R}$ is a set of positive definite matrices. $\mathbf{R}$ being only partly known is typically due to the fact that the cross-covariance $\mathbf{R}_{12}$ of

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_{12} \\ \mathbf{R}_{12}^\top & \mathbf{R}_2 \end{bmatrix},$$

is **unknown**. One way to handle $\mathbf{R} \in \mathfrak{R}$ is to use the *conservative linear unbiased estimator* (CLUE), which is defined below.

Conservative Linear Unbiased Estimator

Given is $\mathbf{y} = \mathbf{Hx} + \mathbf{v}$, where $\mathbf{x}$ is the true state and $\mathbf{v}$ is zero-mean random noise. An estimator on the linear form $\hat{\mathbf{x}} = \mathbf{Ky}$, reporting an error covariance $\mathbf{P}$ of $\hat{\mathbf{x}}$, is called a *conservative linear unbiased estimator* if $\mathrm{E}\,\hat{\mathbf{x}} = \mathbf{x}$ and

$$\mathbf{P} \geq \mathrm{cov}(\hat{\mathbf{x}}),$$

where $\mathrm{cov}(\hat{\mathbf{x}})$ is the true covariance of $\hat{\mathbf{x}} = \mathbf{Ky}$.

Best CLUE

Inspired by the *best linear unbiased estimator* (BLUE), which is the optimal linear unbiased estimator given $\mathbf{y}$ and $\mathbf{R}$ are fully known, we here want to find the *best* CLUE. The best CLUE is given by the following optimization problem

$$\begin{aligned} \mathbf{K}^*, \mathbf{P}^* = \arg\min_{\mathbf{K},\mathbf{P}} \quad & \mathbf{P} \\ \text{subject to} \quad & \mathrm{E}\,\mathbf{Ky} = \mathrm{E}\,\mathbf{K}(\mathbf{Hx}+\mathbf{v}) = \mathbf{x} \\ & \mathbf{P} \geq \mathbf{KR}'\mathbf{K}^\top, \forall \mathbf{R}' \in \mathfrak{R}, \end{aligned} \tag{1}$$

where $\mathrm{E}\,\mathbf{Ky} = \mathbf{x}$ resembles the linear unbiased constraint. The operation $\arg\min \mathbf{P}$ above means minimizing the target $\mathbf{P}$ in the positive semi-definite sense.

Estimation Under Unknown $\mathbf{R}_{12}$

To illustrate the CLUE concept and the effect of neglecting the cross-covariance, consider the following scenario: Let $\mathbf{H} = [\mathbf{I}\ \mathbf{I}]^\top$ and $\mathbf{y} = [\mathbf{y}_1^\top\ \mathbf{y}_2^\top]^\top$ where the covariances of $\mathbf{y}_1$ and $\mathbf{y}_2$, and the cross-covariance are respectively given by

$$\mathbf{R}_1 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{R}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}, \qquad \mathbf{R}_{12} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The covariances from different estimators are given in the the illustration below. It is important to distinguish between the covariance calculated by an estimator (*e.g.*, $\mathbf{P}^{\mathrm{N}}$) and the true covariance (*e.g.*, $\mathrm{cov}(\mathbf{K}^{\mathrm{N}}\mathbf{y})$), where the latter depends on the chosen gain (*e.g.*, $\mathbf{K}^{\mathrm{N}}$). The estimators are defined at the bottom of the page[a][b].

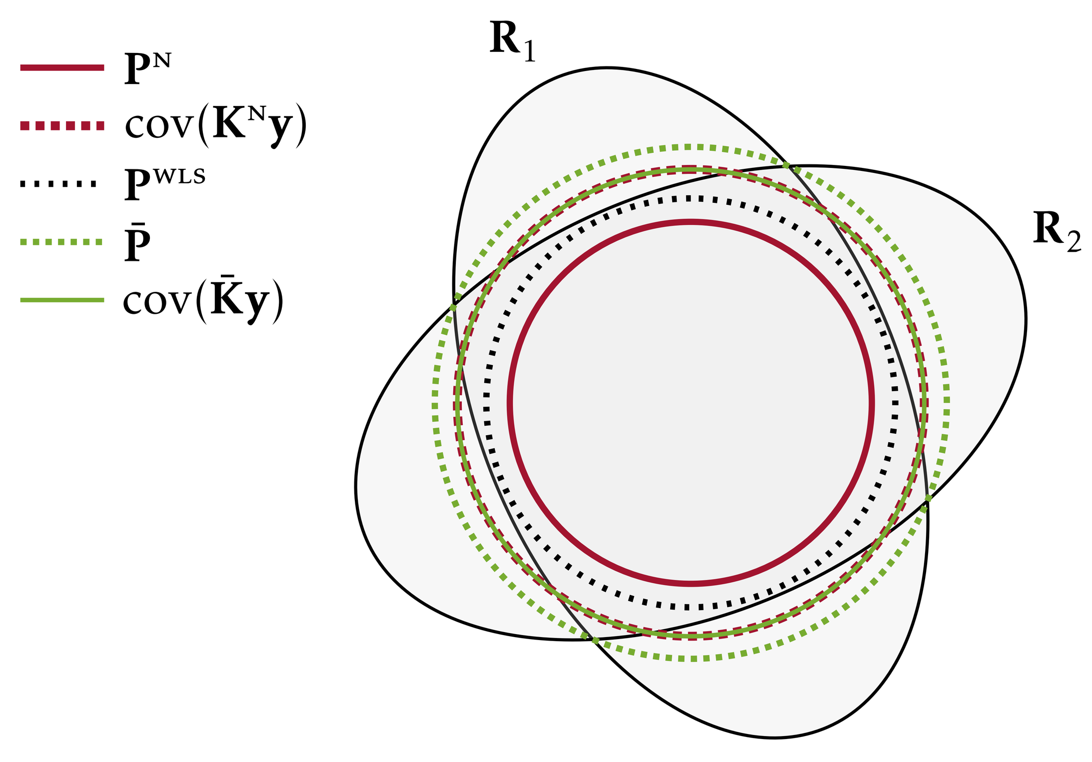

Figure: *$\mathbf{P}^{\mathrm{N}}$ underestimates $\mathrm{cov}(\mathbf{K}^{\mathrm{N}}\mathbf{y})$ and hence is non-conservative. $\bar{\mathbf{P}}$ overestimates $\mathrm{cov}(\bar{\mathbf{K}}\mathbf{y})$ and therefore is conservative.*

Considered Estimators

- Unknown $\mathbf{R}_{12}$: A *naïve* estimator assuming $\mathbf{R}_{12} = \mathbf{0}$ such that $\mathbf{R}' = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix}$ and

$$\begin{aligned} \mathbf{K}^{\mathrm{N}} &= \left(\mathbf{H}^\top(\mathbf{R}')^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^\top(\mathbf{R}')^{-1}, \\ \mathbf{P}^{\mathrm{N}} &= \mathbf{K}^{\mathrm{N}}\mathbf{R}'(\mathbf{K}^{\mathrm{N}})^\top = \left(\mathbf{R}_1^{-1} + \mathbf{R}_2^{-1}\right). \end{aligned}$$

- Known $\mathbf{R}_{12}$: A WLS estimator with $\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_{12} \\ \mathbf{R}_{12}^\top & \mathbf{R}_2 \end{bmatrix}$ and

$$\begin{aligned} \mathbf{K}^{\mathrm{WLS}} &= \left(\mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^\top\mathbf{R}^{-1}, \\ \mathbf{P}^{\mathrm{WLS}} &= \mathbf{K}^{\mathrm{WLS}}\mathbf{R}(\mathbf{K}^{\mathrm{WLS}})^\top = \left(\mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}\right)^{-1}. \end{aligned}$$

- Unknown $\mathbf{R}_{12}$: An estimator with $\bar{\mathbf{R}} = \begin{bmatrix} 2\mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & 2\mathbf{R}_2 \end{bmatrix}$ and

$$\begin{aligned} \bar{\mathbf{K}} &= \left(\mathbf{H}^\top\bar{\mathbf{R}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^\top\bar{\mathbf{R}}^{-1}, \\ \bar{\mathbf{P}} &= \bar{\mathbf{K}}\bar{\mathbf{R}}\bar{\mathbf{K}}^\top = \left(\mathbf{H}^\top\bar{\mathbf{R}}^{-1}\mathbf{H}\right)^{-1}. \end{aligned}$$

[a] All these estimators are linear and unbiased.
[b] WLS: Weighted least squares.

Restricted Best CLUE

Finding a best CLUE, *i.e.*, solving (1) is generally complicated, if not impossible. The problem can be simplified by restricting the problem as follows, which gives a CLUE that is optimal under certain restrictions.

Restricted Best CLUE

Introduce the set $\bar{\mathfrak{R}}$ which contains all $\bar{\mathbf{R}}$ for which $\bar{\mathbf{R}} \geq \mathbf{R}', \forall \mathbf{R}' \in \mathfrak{R}$ is satisfied, and restrict $\mathbf{P}$ to be calculated as $\mathbf{P} = \mathbf{K}\bar{\mathbf{R}}\mathbf{K}^\top$, for some gain $\mathbf{K}$ and *upper bound* $\bar{\mathbf{R}}$. The optimization problem then breaks down to:

- Find the minimum element of $\bar{\mathbf{R}}$, *i.e.*, $\bar{\mathbf{R}}^* = \min \bar{\mathfrak{R}}$ which is the $\bar{\mathbf{R}}^*$ that satisfies $\bar{\mathbf{R}} \geq \bar{\mathbf{R}}^*, \forall \bar{\mathbf{R}} \in \bar{\mathfrak{R}}$.
- Optimize the gain $\mathbf{K}^*$ given $\bar{\mathbf{R}}^*$.

However, finding $\bar{\mathbf{R}}^*$ might still be complicated. Consider the following 2×2 matrix $\mathbf{R} \in \mathfrak{R}$ where the off-diagonal entries are unavailable. $\mathfrak{R}$ is contained in the rectangle of the illustration below:

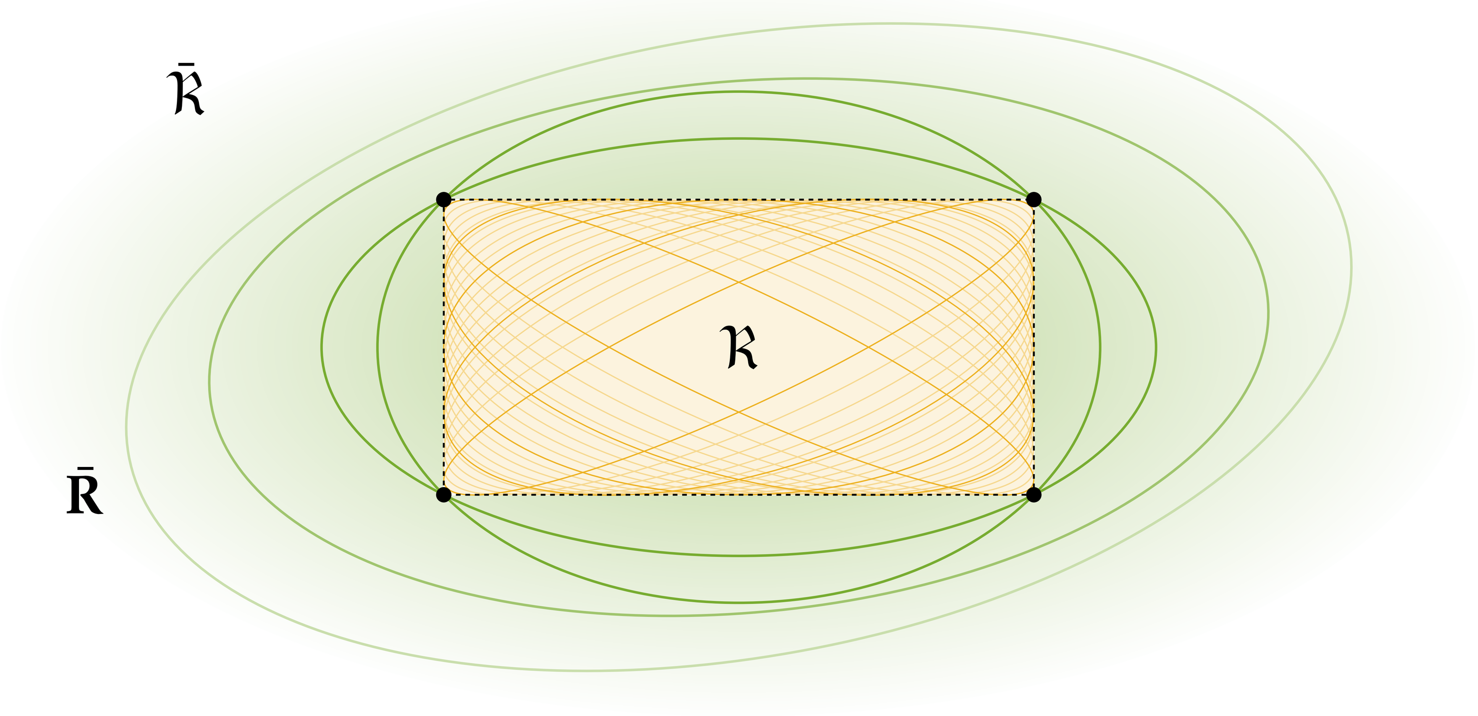

Figure: *Each matrix $\bar{\mathbf{R}} \in \bar{\mathfrak{R}}$ satisfy $\bar{\mathbf{R}} \geq \mathbf{R}', \forall \mathbf{R}' \in \mathfrak{R}$. Finding the minimum $\bar{\mathbf{R}}$ is not possible which can be concluded by looking a those $\bar{\mathbf{R}}$ that tightly encloses the rectangle (where all $\mathbf{R}' \in \mathfrak{R}$ reside) by intersecting the four corners of the rectangle. None of these $\bar{\mathbf{R}}$ will fulfill $\bar{\mathbf{R}}' \geq \bar{\mathbf{R}}, \forall \bar{\mathbf{R}}' \in \bar{\mathfrak{R}}$.*

Possible Resolutions

- Solve $\bar{\mathbf{R}}^* = \min_{\bar{\mathbf{R}} \in \bar{\mathfrak{R}}} J(\bar{\mathbf{R}})$ where $J(\cdot)$ is a loss function.
- Divide the $\bar{\mathbf{R}} \in \bar{\mathfrak{R}}$ into different families according to the parametrization θ they belong to, *e.g.*, $\bar{\mathbf{R}}(\theta) = \begin{bmatrix} \frac{a}{\theta} & 0 \\ 0 & \frac{b}{1-\theta} \end{bmatrix}$.

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