Conservative Linear Unbiased Estimation

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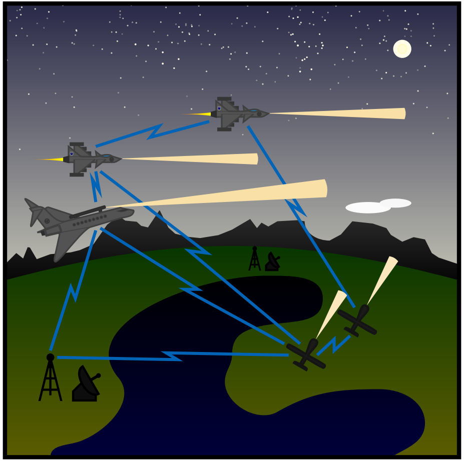
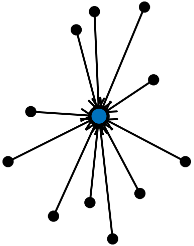
SAAB



Motivating Example

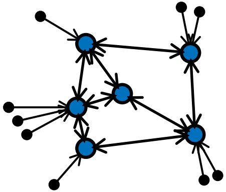
- Heterogenous platforms
- Information extracted by sensors
- Pre-processing of information
  - *Estimates*
- Communication of pre-processed information
- Dependencies: *Cross-correlations*
  - Complicates!
  - Double count information

Centralized Sensor Network



- Optimal information extraction
- Critical nodes
- Combinatorial complexity
- Bandwidth demanding

Decentralized Sensor Network



- Sub-optimal information extraction
- Robust & modular
- Reduced complexity
- Unknown cross-correlations

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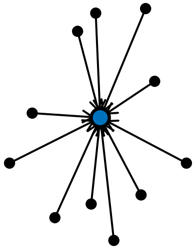

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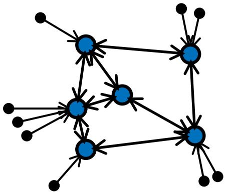

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Unknown Cross-Correlations

Data $\mathbf{y} = \begin{bmatrix}\mathbf{y}_1^\mathsf{T} & \mathbf{y}_2^\mathsf{T}\end{bmatrix}^\mathsf{T} = \mathbf{H}\mathbf{x} + \mathbf{v}$ is given where

$$\operatorname{cov}(\mathbf{y}) = \mathbf{R} = \begin{bmatrix}\mathbf{R}_1 & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_2\end{bmatrix}$$

where $\mathbf{R}_{12} = \mathbf{R}_{21}^\mathsf{T}$ is **unknown**. How to derive an estimate $\hat{\mathbf{x}}$?

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Modelling: $\mathbf{R} \in \mathfrak{R}$ where $\mathfrak{R} = \{\mathbf{R}'_a, \mathbf{R}'_b, \dots\}$

No clue (!) for which $\mathbf{R}' \in \mathfrak{R}$ we have $\mathbf{R} = \mathbf{R}'$

Unknown Cross-Correlations

Problem: Have no **clue** for which $\mathbf{R}' \in \mathfrak{R}$ we have $\mathbf{R} = \mathbf{R}'$

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Solution: Get a **CLUE**

Conservative Linear Unbiased Estimator (CLUE)

An estimator on the form $\hat{\mathbf{x}} = \mathbf{K}\mathbf{y}$ reporting a covariance $\mathbf{P}$ is a *conservative linear unbiased estimator* if $\mathsf{E}\,\hat{\mathbf{x}} = \mathbf{x}$ and

$$\mathbf{P} \succeq \operatorname{cov}(\hat{\mathbf{x}}) = \operatorname{cov}(\mathbf{K}\mathbf{y})$$

Relationship To (Classical) Linear Unbiased Estimation

Classical estimation:

- $\mathfrak{R} = \{\mathbf{R}\}$
- $\hat{\mathbf{x}} = \mathbf{K}\mathbf{y}$ implies covariance is calculated as $\mathbf{P} = \mathbf{K}\mathbf{R}\mathbf{K}^\top$

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Conservative estimation:

- $\mathfrak{R} = \{\mathbf{R}'_a, \mathbf{R}'_b, \dots\}$
- $\hat{\mathbf{x}} = \mathbf{K}\mathbf{y}$ says nothing about how $\mathbf{P}$ is calculated, only that $\mathbf{P} \succeq \operatorname{cov}(\mathbf{K}\mathbf{y})$ which is satisfied if

$$\mathbf{P} \succeq \mathbf{K}\mathbf{R}'\mathbf{K}^\top, \forall \mathbf{R}' \in \mathfrak{R}$$

Optimality

Classical Linear Estimation

$$\begin{aligned}\mathbf{K}^* &= \arg\min_{\mathbf{K}} \quad \mathbf{P}\\ &\text{s.t.} \quad \mathsf{E}\,\mathbf{K}\mathbf{y} = \mathbf{x}\\ &\qquad\quad \mathbf{P} = \mathbf{K}\mathbf{R}\mathbf{K}^\mathsf{T}\end{aligned}$$

Optimal estimator: **BLUE**

$$\hat{\mathbf{x}}^{\text{BLUE}} = \left(\mathbf{H}^\mathsf{T}\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^\mathsf{T}\mathbf{R}^{-1}\mathbf{y}$$

$$\mathbf{P}^{\text{BLUE}} = \left(\mathbf{H}^\mathsf{T}\mathbf{R}^{-1}\mathbf{H}\right)^{-1}$$

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Optimal estimator: **Best CLUE**

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- $\bar{\mathbf{R}} \in \bar{\mathfrak{R}}$ is a *conservative bound*

Restricted Optimality

Classical Linear Estimation

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Conservative Linear Estimation

$$\begin{aligned} \mathbf{K}^*, \bar{\mathbf{R}}^* = \arg\min_{\mathbf{K},\bar{\mathbf{R}}} \quad & \mathbf{P} \\ \text{s.t.} \quad & \mathsf{E}\,\mathbf{K}\mathbf{y} = \mathbf{x} \\ & \mathbf{P} = \mathbf{K}\bar{\mathbf{R}}\mathbf{K}^\mathsf{T}, \bar{\mathbf{R}} \in \bar{\mathfrak{R}} \end{aligned}$$

Optimal estimator: **Restricted Best CLUE**

$$\hat{\mathbf{x}}^* = \left(\mathbf{H}^\mathsf{T}(\bar{\mathbf{R}}^*)^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^\mathsf{T}(\bar{\mathbf{R}}^*)^{-1}\mathbf{y}$$

$$\mathbf{P}^* = \left(\mathbf{H}^\mathsf{T}(\bar{\mathbf{R}}^*)^{-1}\mathbf{H}\right)^{-1}$$

Restricted Best CLUE

Problem boils down to:

Find the *minimum conservative bound* $\bar{\mathbf{R}}^*$

where a conservative bound $\bar{\mathbf{R}}$ has the property

$$\bar{\mathbf{R}} \succeq \mathbf{R}', \forall \mathbf{R}' \in \mathfrak{R}$$

Minimum Conservative Bounds

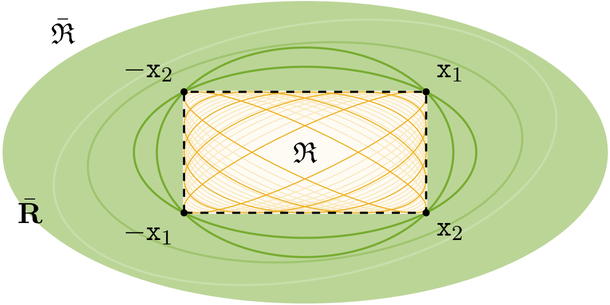
Assume:

- $\mathbf{R} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$
- a, d known
- b unknown

Points:

$$\mathrm{x}_1 = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix}$$

$$\mathrm{x}_2 = \begin{bmatrix} \sqrt{a} \\ -\sqrt{b} \end{bmatrix}$$



Minimum Conservative Bounds

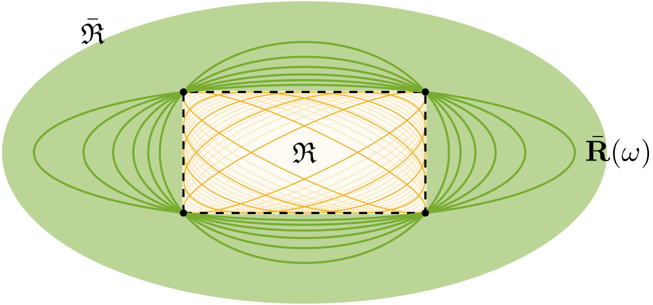
Problem:

- Incomparable matrices

Solutions:

- Loss function $J(\mathbf{P})$
- Parametrize $\bar{\mathbf{R}}$

Figure:
$\bar{\mathbf{R}}(\omega) = \frac{a}{\omega} + \frac{b}{1-\omega}$
$\omega \in (0, 1)$



Thanks for listening!

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