RL workshop March 2021
System Identification: An Overview

Build mathematical models from observed input and output signals

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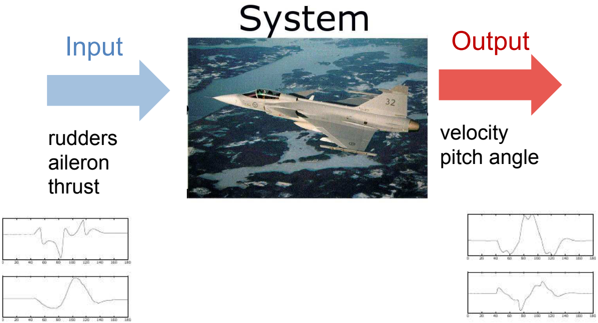
An Introductory Example: System

The System

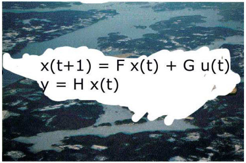


An Introductory Example 2: Model

The Model

u
Input

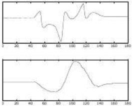
MODEL



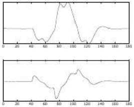
y
Output

rudders
aileron
thrust

velocity
pitch angle

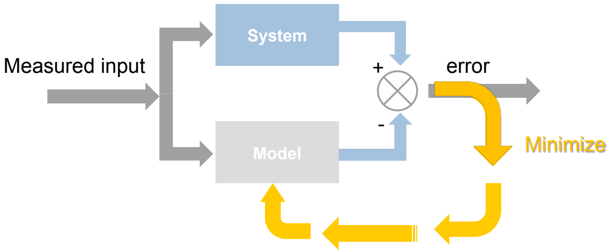


u, y: measured time or frequency domain signals

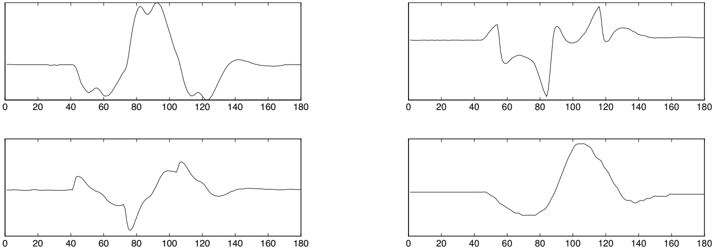


An Introductory Example 3: Model Fitting

The System and the Model



Data from the Gripen Aircraft



Output: Pitch
Inputs: Canard, Elevator, Leading Edge Flap

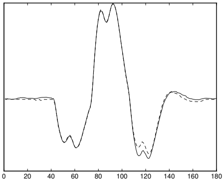
- How do the control surface angles affect the pitch rate?

Using All Inputs

u_1 canard angle; u_2 Elevator angle; u_3 Leading edge flap;

$$
\begin{aligned}
y(t) = & -a_1 y(t-T) - a_2 y(t-2T) - a_3 y(t-3T) - a_4 y(t-4T) \\
& + b_1^1 u_1(t-T) + \ldots + b_1^4 u_1(t-4T) \\
& + b_2^1 u_2(t-T) + \ldots + b_1^3 u_3(t-T) + \ldots + b_4^3 u_3(t-4T)
\end{aligned}
$$

Estimate 16 parameter using half of the data record – Simulate the model using the whole data record.

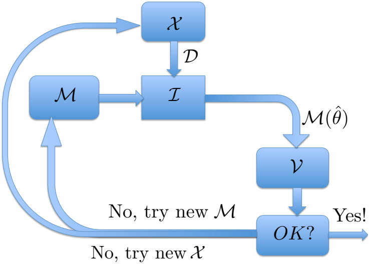


Dashed line: Measured Pitch rate. Solid line: The pitch rate according to the model.
First half estimation data - second half validation data.

System Identification: Issues

- Select a class of candidate models
- Select a member in this class using the observed data
- Evaluate the quality of the obtained model
- Design the experiment so that the model will be "good".

The System Identification Flow



$\mathcal{X}$: The Experiment
$\mathcal{D}$: The Measured Data
$\mathcal{M}$: The Model Set
$\mathcal{I}$: The Identification Method
$\mathcal{V}$: The Validation Procedure

Models: General Aspects for Dynamical Systems

- A model is a mathematical expression that describes the connections between measured inputs and outputs, and possibly related noise sequences.
- They can come in many different forms
- The models are labeled with a parameter vector θ
- A common framework is to describe the model as a predictor of the next output, based on observations of past input-output data.

  Observed input–output (u, y) data up to time t: Z^t

  Model described by predictor: $\mathcal{M}(\theta) : \hat{y}(t|\theta) = g(t, \theta, Z^{t-1})$.

Estimation

If a model, $\hat{y}(t|\theta)$, essentially is a predictor of the next output, is is natural to evaluate its quality by assessing how well it predicts:
Form the *Prediction error* and measure its size:

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t|\theta), \quad \ell(\varepsilon(t,\theta)) = \varepsilon^2(t,\theta)$$

How has it performed historically?

$$V_N(\theta) = \sum_{t=1}^{N} \ell(\varepsilon(t,\theta))$$

Which model in the structure performed best?

$$\hat{\theta}_N = \arg\min_{\theta \in D_{\mathcal{M}}} V_N(\theta)$$

Often coincides with the Maximum Likelihood Estimate.

Linear Regressions

The linear regression:

$$y(t) = \varphi^T(t)\theta + e(t)$$

$y(t)$ and $\varphi(t)$ known/measured at time t. Find a good value of θ!
This covers many useful systems and signals models:

- AR: $\varphi^T(t) = [-y(t-1)\ldots - y(t-n)]$
- ARX:

$$\varphi^T(t) = [-y(t-1)\ldots - y(t-n), u(t-1)\ldots u(t-m)]$$

- "Semi-physical", non-linear models:

$$\begin{aligned} y(t) &= a_1 y^3(t-1) + a_2 y(t-1)u_1(t-1) + \\ &\quad + a_3 \log u_2(t-2) \end{aligned}$$

The (Recursive) Least Squares Estimate

$$\hat{\theta}(t) = \arg\min \sum_{j=1}^{t} (y(j) - \varphi^T(j)\theta)^2 = \left[\sum_{j=1}^{t} \varphi(j)\varphi^T(j)\right]^{-1} \sum_{j=1}^{t} y(j)\varphi(j)$$
$$= R^{-1}(t) f(t)$$

... can be exactly rewritten

$$\hat{\theta}(t) = \hat{\theta}(t-1) + R^{-1}(t)\varphi(t)\varepsilon(t)$$
$$\varepsilon(t) = y(t) - \varphi^T(t)\hat{\theta}(t-1)$$
$$R(t) = R(t-1) + \varphi(t)\varphi^T(t)$$

Check Algebra!

$$
\begin{aligned}
\hat{\theta}(t) &= R^{-1}(t)f(t) = R^{-1}(t)(f(t-1) + \varphi(t)y(t)) \\
&= R^{-1}(t)(R(t-1)\hat{\theta}(t-1) + \varphi(t)y(t)) \\
&= R^{-1}(t)[(R(t) - \varphi(t)\varphi^T(t))\hat{\theta}(t-1) + \varphi(t)y(t)] \\
&= \hat{\theta}(t-1) + R^{-1}(t)\varphi(t)[y(t) - \varphi^T(t)\hat{\theta}(t-1)])
\end{aligned}
$$

Note that

$$
\begin{aligned}
P(t) &= R^{-1}(t) = [R(t-1) + \varphi(t)\varphi^T(t)]^{-1} \\
P(t) &= P(t-1) + \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{1 + \varphi^T(t)P(t-1)\varphi(t)}
\end{aligned}
$$

So ...

Recursive Least Squares

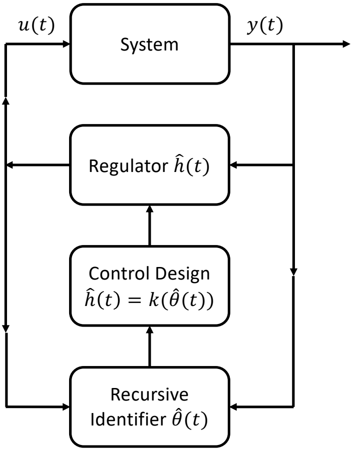
$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\varphi(t)\varepsilon(t)$$
$$\varepsilon(t) = y(t) - \varphi^T(t)\hat{\theta}(t-1)$$
$$P(t) = P(t-1) + \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{1+\varphi^T(t)P(t-1)\varphi(t)}$$

Note that the updating is driven by $\varphi(t)\varepsilon(t) = -\frac{1}{2}\frac{d}{d\theta}[\varepsilon^2(t)]$ (the reward for a good model!

Compare with Gradient Policy RL!

Adaptive Control

Use a recursive estimator to build a system model at all times.
Compute a controller k based on the current model!



Example (used in RL) Measure all states

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = x(t) + e(t)$$

Example:

$$A = \begin{bmatrix} 1.01 & 0.01 & 0 \\ 0.01 & 1.01 & 0.01 \\ 0 & 0.01 & 1.01 \end{bmatrix} \qquad B = I$$

Identification code: [Matlab System identification Toolbox]

```
ms = idss(A,B,C,D)
m0=idss(rand(3,3),rand(3,3),eye(3,3),zeros(3,1))
m0.Structure.C.Free = zeros(3,3)    [ % C fixed to identity]
m = ssest(data,m0,ssestOptions,'DisturbanceModel', 'est');
A = m.A, B = m.B;
```

Identification Result

Note that system is unstable; it must be run under stabilizing feedback!

Test: 100 observations, Additive observation noise with variance 1 in each channel: Gives the model (cf true values)

$$\hat{A} = \begin{bmatrix} 1.0016 & 0.0065 & 0.0076 \\ 0.0267 & 1.0192 & 0.0101 \\ 0.0370 & 0.0419 & 0.9672 \end{bmatrix} \quad A = \begin{bmatrix} 1.01 & 0.01 & 0 \\ 0.01 & 1.01 & 0.01 \\ 0 & 0.01 & 1.01 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 1.0101 & 0.0140 & -0.0094 \\ 0.0058 & 0.9899 & -0.0120 \\ 0.0121 & 0.0194 & 0.9607 \end{bmatrix} \quad B = I$$

Summary

- Traditional Control approach to RL: Build a model and use that for control design. ("Model Building RL").
- System identification is a general and versatile tool to build models from data.
- Discuss next time (April 6) Pro's and con's of this traditional approach compared to the new techniques.