Conservative Linear Unbiased Estimation

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Considered Problem

Assume a linear regression

$$y = Hx^0 + v \qquad \text{cov}(y) = R^0 \in \mathcal{A}$$

where the true state $x^0 \in \mathbf{R}^n$ is to be estimated given noisy data $y \in \mathbf{R}^m$ corrupted by zeromean noise v. It is assumed that R^0 is *unknown* but is an element of a known set $\mathcal{A}$.

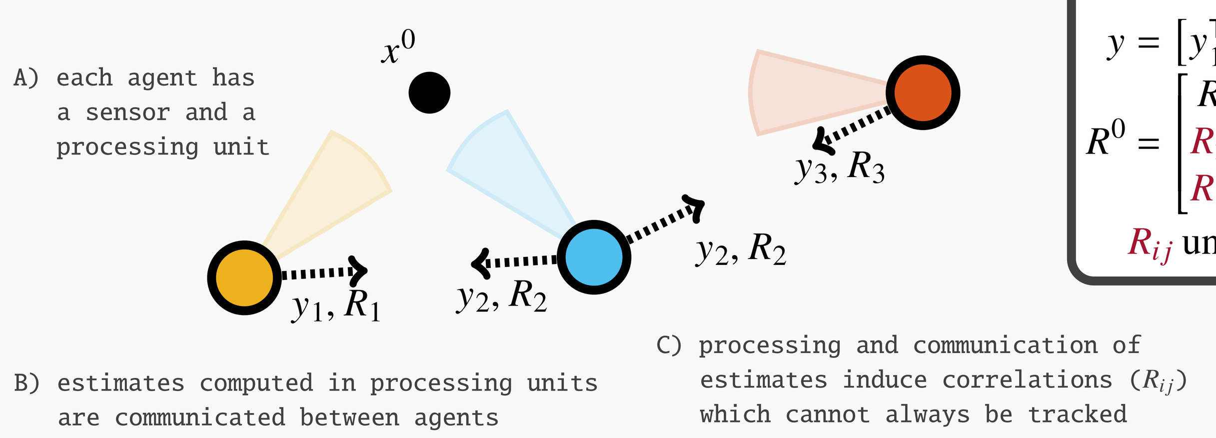
Problem Statement

Given $y = Hx^0 + v$ and $\mathcal{A}$, compute an estimate $\hat{x}$ of x^0 with covariance P, where P **is as small as possible but not smaller** than the true covariance $\text{cov}(\hat{x})$ of $\hat{x}$.

It is assumed $\mathcal{A} \subset \mathbb{S}^m_{++}$ where $\mathbb{S}^m_{++}$ is the set of all $m \times m$ symmetric positive definite matrices.

Background

Having only $R^0 \in \mathcal{A}$ is a typical issue in *decentralized estimation*. Let (y_i, R_i) be the ith estimate of x^0:



Full structure

$$y = [y_1^\top \; y_2^\top \; y_3^\top]^\top$$

$$R^0 = \begin{bmatrix} R_1 & R_{12} & R_{13} \\ R_{21} & R_2 & R_{23} \\ R_{31} & R_{32} & R_3 \end{bmatrix}$$

R_{ij} unknown

Proposed Framework

An estimator $(\hat{x}, P)$ is a *conservative linear unbiased estimator* (CLUE) if it has the following **properties**:

$$\underbrace{\hat{x} = Ky}_{linear} \qquad \underbrace{KH = I}_{unbiased} \qquad \underbrace{P \succeq KRK^\top, \forall R \in \mathcal{A}}_{conservative}$$

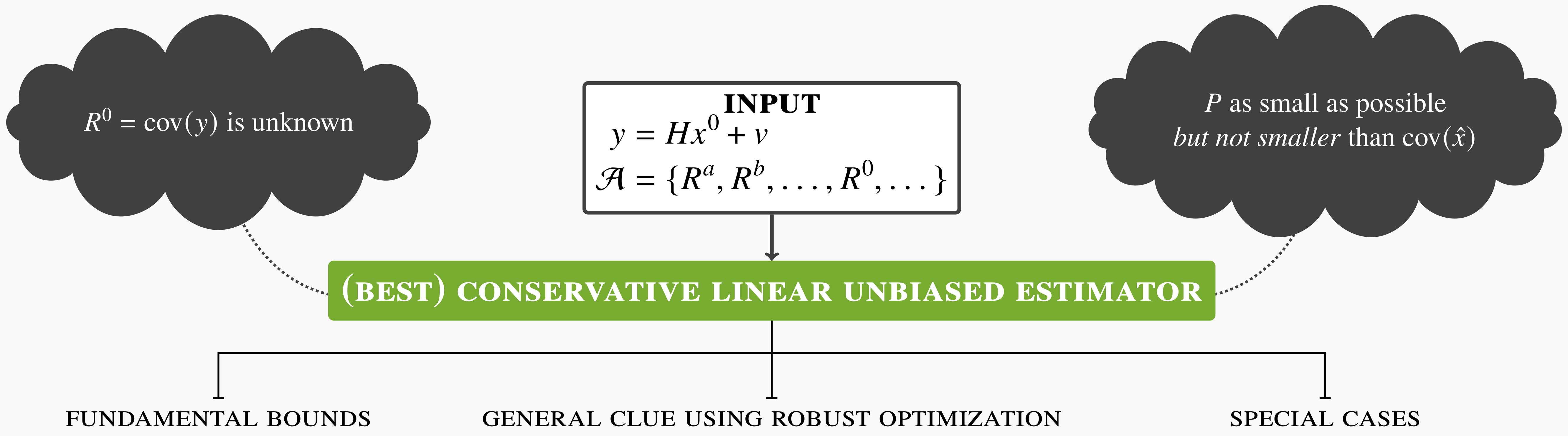
An **optimal CLUE** is defined as follows:

Best CLUE

Let J be a loss function. An estimator reporting $\hat{x}^\star = K^\star y$ and $P^\star$ is called a *best CLUE* if $(K^\star, P^\star)$ is the solution to:

$$\begin{aligned} &\underset{K,P}{\text{minimize}} && J(P) \\ &\text{subject to} && KH = I \\ &&& P \succeq KRK^\top, \forall R \in \mathcal{A}. \end{aligned}$$

The CLUE Framework



Fundamental Bounds

It can be shown that $P^u \succeq P^\star \succeq P_l$, where P_u is constructed from C such that $C \succeq R, \forall R \in \mathcal{A}$ and P_l is the smallest covariance larger than the covariance of all possible *best linear unbiased estimators* (BLUE).

Benefits of bounds:

- *Guideline for system design*
- *Problem simplification*

Let $C \succeq R, \forall R \in \mathcal{A}$, then:

$$K_u = (H^\top C^{-1} H)^{-1} H^\top C^{-1}$$
$$P_u = (H^\top C^{-1} H)^{-1}$$

is an **upper bound** on a best CLUE

$$P_u \succeq P^\star \succeq P_l$$

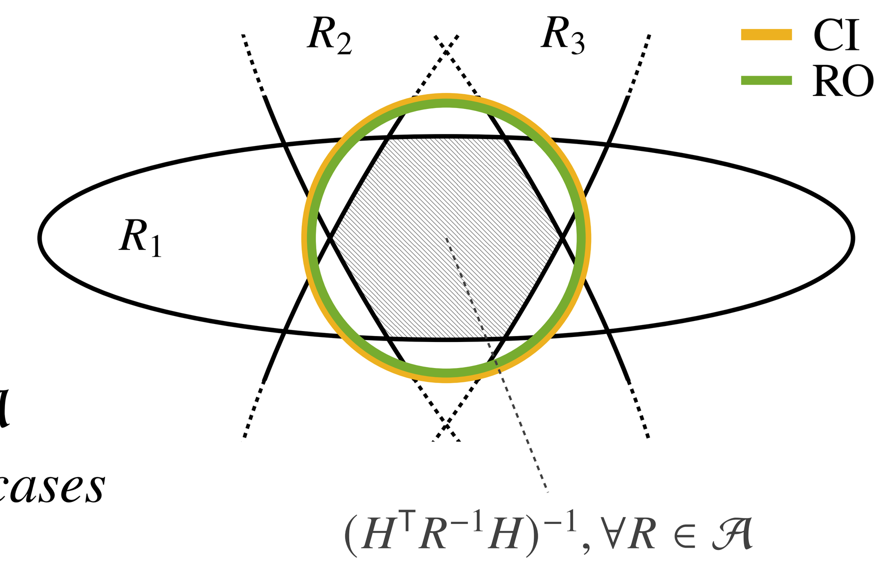
A minimal P_l larger than all possible BLUE is the solution to:

$$\begin{aligned} &\underset{P}{\text{minimize}} && J(P) \\ &\text{subject to} && P \succeq (H^\top R^{-1} H)^{-1}, \forall R \in \mathcal{A} \end{aligned}$$

and is a **lower bound** on a best CLUE.

General CLUE Using Robust Optimization

It can be shown that the uncertainty imposed by $R^0 \in \mathcal{A}$ fits into an established optimization paradigm called *robust optimization* (RO).



Benefits of CLUE using RO:

- *Solving problems with arbitrary* $\mathcal{A}$
- *Improved performance in certain cases*

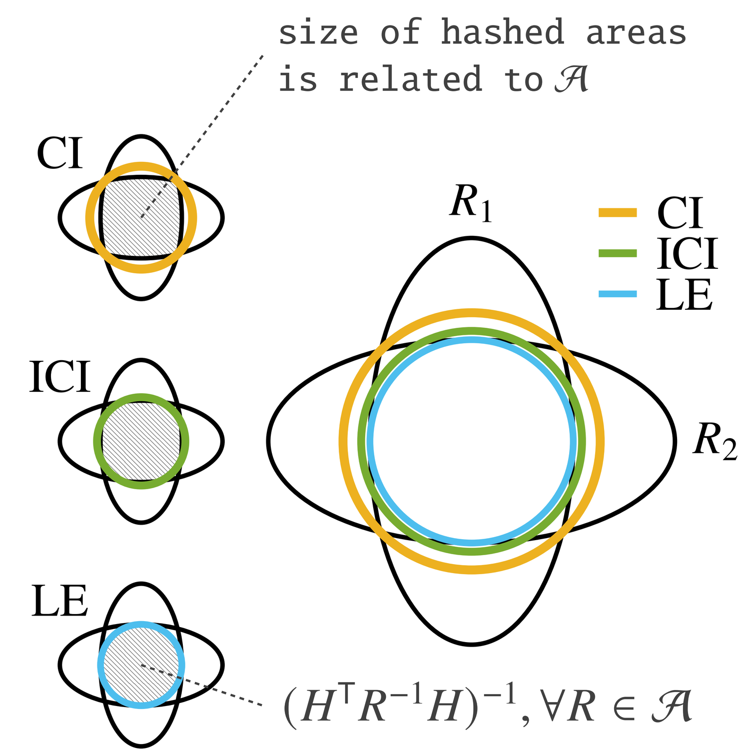
Special Cases

Observations:

- Performance depends on $\mathcal{A}$
- More knowledge about the unknown parts of R^0 means smaller $\mathcal{A}$
- Smaller $\mathcal{A}$ means possibly smaller $P^\star$

Several conservative estimation methods exists, and can be shown to be **special cases of a best CLUE** under different assumptions on $\mathcal{A}$:

- *Covariance intersection* (CI)
- *Inverse covariance intersection* (ICI)
- *Largest ellipsoid* (LE) method



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