Algebraic aspects of Riemannian geometry

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Aim of todays lecture

- Introduce the general idea of identifying geometric spaces with the corresponding (commutative) function algebras.
- Give an idea of what noncommutative geometry is about.
- Recall how geometrical objects may be though of in an algebraic way.
- Generalization of geometric concepts to noncommutative algebras.

General idea

To study a geometric space through the algebra of functions from the space to a field $\mathbb{K}$. That is, study functions instead of points.

Why non-commutative geometry?

- Quantum gravity.
- Just as algebraic geometry can be seen as a way to study commutative rings, non-commutative geometry can be used to study non-commutative rings.
- There are spaces (e.g. the space of leaves of a foliation) which have no interesting measurable (real-valued) functions. However, they exhibit interesting operator-valued functions.
- Deep relation to operator algebras.
- Index theory / K-theory. The generalizations of the Atiyah-Singer index theorem leads in a very natural way to non-commutative geometry.

Quantum gravity and non-commutative geometry

Why is non-commutative geometry important for physics, in particular quantum gravity?

Most physicists believe that non-commutative geometry is necessary (in some form) to reconcile the two most outstanding physical theories of the 20th century: quantum mechanics and general relativity.

At very small distances (10^{-35}m, the Planck length) our "normal" concepts of length and geometry break down as measurements are no longer possible. To measure distances one needs a photon with a wave-length which is shorter than the distance one wants to measure. To measure something at the Planck length, the energy of the photon would be high enough to create a black hole, thus making all measurements even theoretically impossible.

Hence, if one wants to understand a deeply geometric physical theory – General relativity – one has to make sense of geometry in noncommutative algebras.

General ideas / Rules of the game

1. Study the algebra of functions on a geometric space and try to express geometric quantities in algebraic terms.
2. Investigate to what extent these concepts make sense in an arbitrary commutative algebra.
3. Can the above concepts be implemented in a non-commutative algebra?
4. Study non-commutative algebras which allow for the implementation of geometric concepts.

The (commutative) algebra of functions

Let $\mathbb{K}$ be a field ($\mathbb{R}$ or $\mathbb{C}$). The set of $\mathbb{K}$-valued functions on a space X is a ring:

- $(f+g)(x)=f(x)+g(x)$
- $(fg)(x)=f(x)g(x)$
- Unit: constant function $f(x)=1$
- Zero: constant function $f(x)=0$

Moreover, it is a vector space over $\mathbb{K}$, i.e a $\mathbb{K}$-algebra.

Points as a dual objects

Given an algebra of functions $\mathcal{A}$, one may consider a point x to be a linear functional $x : \mathcal{A} \to \mathbb{K}$:

$$x(f) = f(x).$$

Furthermore, in the commutative case, the point is considered to be an algebra homomorphism, e.g.

$$x(f \cdot g) = x(f)x(g) = f(x)g(x).$$

Hence, there is a *duality* between functions and points (which is, of course, at the heart of algebraic geometry). However, we shall mainly not be interested in recovering the points from the algebra, but rather to forget about the points all together.

Example – The sphere

For instance, instead of directly considering the sphere,

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 1 = 0\},$$

one studies the ring of polynomial functions from the sphere to $\mathbb{R}$

$$\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$$

where $(x^2 + y^2 + z^2 - 1)$ denotes the ideal generated by the element $x^2 + y^2 + z^2 - 1$. That is, the polynomial $x^2 + y^2 + z^2 - 1$ is identified with 0.

One can study the function algebra in many different settings:

- Polynomial functions $\rightarrow$ Algebraic Geometry,
- Continuous functions on a topological space,
- Smooth functions on a differentiable manifold,
- ...

In each case, one tries to formulate geometric properties in terms of algebraic properties of the function algebra, such that one can do geometry in a purely algebraic way.

Once this is done, one often tries to abstract the algebra from the underlying space. That is, one tries to find a class of algebras for which the geometric concepts make sense.

Let us make one of the dualities above more precise.

$*$-algebras

Much of noncommutative geometry is done in the context of $*$-algebras, which are associative algebras together with an involution, playing the role of a complex conjugation.

Definition

A $*$-algebra $(\mathcal{A}, *)$ is an associative algebra (over $\mathbb{C}$) such that $* : \mathcal{A} \to \mathcal{A}$ satisfies

$$\begin{aligned}
(a^*)^* &= a \\
(a+b)^* &= a^* + b^* \\
(ab)^* &= b^* a^* \\
(\lambda a)^* &= \bar{\lambda} a^*
\end{aligned}$$

for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

For instance, matrix algebras over $\mathbb{C}$ together with hermitian conjugate.

C^*-algebras

The "analytic" version of $*$-algebras are C^*-algebras which are closely related to operator algebras.

Definition

A Banach algebra $(\mathcal{A}, ||\cdot||)$ is an associative algebra (over $\mathbb{C}$) which is also a Banach space, such that $||ab|| \leq ||a||\,||b||$ for all $a, b \in \mathcal{A}$.

Definition

A C^*-algebra $(\mathcal{A}, ||\cdot||, *)$ is a Banach algebra $(\mathcal{A}, ||\cdot||)$ and a $*$-algebra $(\mathcal{A}, *)$ such that $||x^*x|| = ||x||\,||x^*||$ (C^*-condition).

- Continuous $\mathbb{C}$-valued functions on a top. space, vanishing at infinity.
- Bounded operators on a Hilbert space.

The (commutative) Gelfand-Naimark theorem

The algebra of continuous $\mathbb{C}$-valued functions that vanish at infinity on a (loc. compact, Hausdorff) topological space is a commutative C^*-algebra.

A commutative C^*-algebra A gives rise to a topological space $X(A)$ by considering the space of $*$-homomorphism from the algebra to $\mathbb{C}$.

Theorem (Gelfand-Naimark)

$$A \simeq C_0\big(X(A)\big)$$

The theorem shows that there is a categorical equivalence between the category of (locally compact) topological spaces and the category of commutative C^*-algebras.

One can also show that the topological space is compact if and only if the corresponding C^*-algebra is unital.

Noncommutative topology

Hence, the study of topological spaces is *the same thing* as the study of commutative C^*-algebras.

It seems quite natural to think about noncommutative C^*-algebras as "noncommutative topology".

Of course, this is just a way of thinking and nothing prevents us from studying general associative algebras by geometric means.

Building up a dictionary

Now, how to we translate geometric concepts to a (noncommutative) algebraic setting?

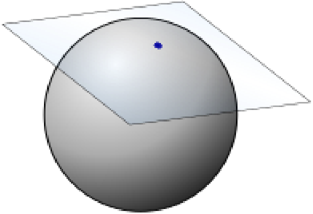
We have already seen that

$$\text{Algebra of functions} \leftrightarrow \text{Associative algebra over } \mathbb{C}$$
$$\text{Compact space} \leftrightarrow \text{Unital algebra}$$

Let us now take a look at how one can introduce other geometric notions in an algebraic language.

Vector bundles

- A vector bundle is a smooth assignment of a vector space to each point of a manifold M.
- It can be as simple as the Cartesian product $M \times \mathbb{R}^n$ (trivial bundle), but there are non-trivial vector bundles.
- An example of a vector bundle is the tangent bundle of a manifold, which assigns the vector space of tangent vectors to each point.
- For instance, the tangent bundle of the sphere is not a trivial bundle, but the tangent bundle of a torus is a trivial bundle.



Sections of vector bundles

- Given a vector bundle E, a smooth choice of a vector for each point is called a section of the vector bundle.
- Given two vector fields $X, Y : M \to E$ one can add them, and multiply by functions

$$
\begin{aligned}
(X + Y)(p) &= X(p) + Y(p) \\
(fX)(p) &= f(p)X(p)
\end{aligned}
$$

  for $f \in C^\infty(M)$ and $p \in M$.
- That is, the space of sections of a vector bundle is a module over the algebra of functions $C^\infty(M)$.

Vector bundles and projective modules

Because of the following theorem by Serre and Swan (here in the form of Swan) one has a good algebraic notion of a vector bundle.

Theorem (R. G. Swan)

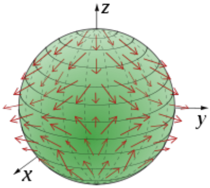
Let X be a compact Hausdorff space, and let $C(X)$ be the ring of continuous functions from X to $\mathbb{R}$. A $C(X)$-module P is isomorphic to a module of sections of a vector bundle if and only if it is a finitely generated projective module.

Hence, one simply defines a vector bundle over an arbitrary (noncommutative) algebra $\mathcal{A}$ as a finitely generated projective $\mathcal{A}$-module.

Vector bundle $\leftrightarrow$ Finitely generated projective module

Vector fields

A section of the tangent bundle is called a vector field. It is an assignment of a tangent vector to each point of the manifold.



In local coordinates, a vector field can be written as

$$X = X^i \frac{\partial}{\partial x^i}$$

and act as a derivative

$$X(f) = X^i \frac{\partial f}{\partial x^i}$$

for $f \in C^\infty(M)$. Let us denote the space of vector fields by $\mathcal{X}(M)$.

Vector fields and derivations

So, a vector field is a derivation of the algebra $C^\infty(M)$. Recall that a derivation ∂ of a $\mathbb{K}$-algebra $\mathcal{A}$ is a $\mathbb{K}$-linear map $\partial : \mathcal{A} \to \mathcal{A}$ such that

$$\partial(ab) = a(\partial b) + (\partial a)b.$$

In fact, one can show that the set of derivations of $C^\infty(M)$ is in one to one correspondence with the set of vector fields.

$$\text{Vector fields} \leftrightarrow \text{Derivations}$$

Note that for a commutative algebra, the set of derivations is a module over the algebra. This is **not** true for a noncommutative algebra. However, the set of derivations is still a module over the center of the algebra.

(Computation on the board.)

Thus, if one is looking for a module to serve as vector fields in a noncommutative setting, the set of derivations is not suitable.

Differential forms

- Differential forms are dual objects to vector fields.
- At each point a differential n-form $\omega \in \Omega^n(M)$ is a multi linear alternating map of n tangent vectors to a number.
- Globally, a differential n-form is a multi linear alternating map

$$\omega : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M)$$

- The set of n-forms $\Omega^n(M)$ is a module over $C^\infty(M)$.

$$\begin{aligned}
(\omega + \eta)(X_1, \dots, X_n) &= \omega(X_1, \dots, X_n) + \eta(X_1, \dots, X_n) \\
(f\omega)(X_1, \dots, X_n) &= f\big(\omega(X_1, \dots, X_n)\big)
\end{aligned}$$

- A n-form ω and a m-form η can be multiplied to give a $(m+n)$-form $\omega \wedge \eta$.
- The exterior derivative $d : \Omega^n(M) \to \Omega^{n+1}(M)$. In particular, $df \in \Omega^1(M)$ for $f \in C^\infty(M)$.

Differential graded algebra

The direct sum of differential forms of all degrees

$$\Omega(M) = \bigoplus_{k\geq 0} \Omega^k(M)$$

(recall that $\Omega^k(M) = 0$ for $k > \dim(M)$) is a differential $\mathbb{N}$-graded algebra, graded by the degree of the form, with $\Omega^0(M) = C^\infty(M)$.

For $\omega \in \Omega^n$ and $\eta \in \Omega^m(M)$:

$$\begin{aligned}
&\omega \wedge \eta \in \Omega^{n+m}(M) \\
&d\omega \in \Omega^{n+1}(M) \\
&d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^n \omega \wedge (d\eta)
\end{aligned}$$

Note: In a differential graded algebra there is in general **no** relation like $\omega \wedge \eta = (-1)^{mn} \eta \wedge \omega$. If this is the case, the algebra is called graded commutative.

Differential calculus

- Thus, if one wants data corresponding to a differentiable manifold one should, apart from choosing an algebra $\mathcal{A}$, provide some kind of "differentiable structure".
- In the spirit of Alain Connes (Fields medallist and the "father" of noncommutative geometry) one chooses a representation of the algebra on a Hilbert space together with a "Dirac operator" acting on the space. (This also gives metric information.)
- Another way of doing this is to choose a differential graded algebra Ω such that $\Omega^0 = \mathcal{A}$.
- Yet another way is to choose a distinguished set of derivations on the algebra, defining the calculus.
- In noncommutative geometry, we have to live with the fact that there are many possible choices of differential calculus over an algebra.

Derivation based differential calculus

- In this approach (pioneered by Michel Dubois-Violette), one starts by choosing an algebra $\mathcal{A}$ together with a Lie algebra $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$.
- Is $\mathfrak{g} = \mathrm{Der}(\mathcal{A})$ a canonical choice? Not always, a noncommutative algebra has plenty of inner derivations $\partial(a) = [a, D]$ for some $D \in \mathcal{A}$.
- For several reasons, one is usually more interested in outer derivations.
- Now, let us start with the pair $(\mathcal{A}, \mathfrak{g})$ and build a differential graded algebra.
- The algebra $\mathcal{A}$ correspond to the "functions" and the Lie algebra $\mathfrak{g}$ correspond to the "vector fields". The differential graded algebra will correspond to the "differential forms".

Thus, given $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$, one defines $\bar{\Omega}^k_{\mathfrak{g}}$ to be the set of $Z(\mathcal{A})$-multilinear alternating maps ($Z(\mathcal{A})$ =center of $\mathcal{A}$)

$$\omega : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k} \to \mathcal{A},$$

and one gives $\bar{\Omega}^k_{\mathfrak{g}}$ the structure of a $\mathcal{A}$-bimodule by setting

$$\begin{aligned}(a\omega)(\partial_1, \ldots, \partial_k) &= a\omega(\partial_1, \ldots, \partial_k)\\ (\omega a)(\partial_1, \ldots, \partial_k) &= \omega(\partial_1, \ldots, \partial_k)a\end{aligned}$$

for $a \in \mathcal{A}$, $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ and $\partial_1, \ldots, \partial_k \in \mathfrak{g}$.
Furthermore, for $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ and $\tau \in \bar{\Omega}^l_{\mathfrak{g}}$ one defines $\omega\tau \in \bar{\Omega}^{k+l}_{\mathfrak{g}}$ as

$$\begin{aligned}&(\omega\tau)(\partial_1, \ldots, \partial_{k+l})\\ &\qquad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega(\partial_{\sigma(1)}, \ldots, \partial_{\sigma(k)})\tau(\partial_{\sigma(k+1)}, \ldots, \partial_{\sigma(k+l)}),\end{aligned}$$

where S_N denotes the symmetric group on N letters.

For $a \in \mathcal{A}$ one defines $d_0 : \mathcal{A} = \bar{\Omega}^0_{\mathfrak{g}} \to \bar{\Omega}^1_{\mathfrak{g}}$ as

$$(d_0 a)(\partial) = \partial a$$

and for $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ (for $k \geq 1$) one defines $d_k : \bar{\Omega}^k_{\mathfrak{g}} \to \bar{\Omega}^{k+1}_{\mathfrak{g}}$ by

$$\begin{aligned} d_k\omega(\partial_0, \ldots, \partial_k) &= \sum_{i=0}^{k} (-1)^i \partial_i \big(\omega(\partial_0, \ldots, \hat{\partial}_i, \ldots, \partial_k)\big) \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega\big([\partial_i, \partial_j], \partial_0, \ldots, \hat{\partial}_i, \ldots, \hat{\partial}_j, \ldots, \partial_k\big), \end{aligned}$$

satisfying $d_{k+1} d_k = 0$, where $\hat{\partial}_i$ denotes the omission of ∂_i in the argument. When there is no risk for confusion, we shall omit the index k and simply write $d : \bar{\Omega}^k_{\mathfrak{g}} \to \bar{\Omega}^{k+1}_{\mathfrak{g}}$.

Since $d^2 = 0$ there is a natural cohomology theory

$$H^k(\Omega_{\mathfrak{g}}) = \ker(d_k) / \operatorname{im}(d_{k-1})$$

Differential calculus over $\mathcal{A}$

Finally, we set

$$\bar{\Omega}_{\mathfrak{g}} = \bigoplus_{k\geq 0} \bar{\Omega}^k_{\mathfrak{g}}$$

and note that $(\bar{\Omega}_{\mathfrak{g}}, d)$ is a differential graded algebra. In this way, we have constructed a differential calculus over an algebra $\mathcal{A}$ from a choice of derivations.

$\bar{\Omega}^1_{\mathfrak{g}}$ is an $\mathcal{A}$-bimodule, which we think of as (sections of) the cotangent bundle – differential 1-forms.

In many cases, one is interested in the differential graded subalgebra $\Omega_{\mathfrak{g}} \subseteq \bar{\Omega}_{\mathfrak{g}}$ generated by the algebra $\mathcal{A}$. That is, $\omega \in \Omega^k_{\mathfrak{g}}$ can be written as

$$\omega = a_0 da_1 da_2 \cdots da_k$$

for $a_0, \ldots, a_k \in \mathcal{A}$.

A few examples of computations on the board.

The noncommutative torus

Let T^2_θ be the unital associative $*$-algebra generated by U, V such that

$$\begin{aligned} VU &= e^{i2\pi\theta} UV \\ UU^* &= U^*U = \mathbb{1} = VV^* = V^*V \end{aligned}$$

One should think of $U \sim e^{i\varphi_1}$ and $V \sim e^{i\varphi_2}$ when relating to the torus. There are two "natural" outer derivations corresponding to $\frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2}$

$$\begin{aligned} \partial_1 U &= iU \qquad & \partial_1 V &= 0 \\ \partial_2 U &= 0 \qquad & \partial_2 V &= iV \end{aligned}$$

satisfying $[\partial_1, \partial_2] = 0$. Let $\mathfrak{g}$ denote the Lie algebra generated by ∂_1, ∂_2.

(Differential graded algebra of T^2_θ on the board.)

Connections and curvature

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Der}(\mathcal{A})$.

Definition

Let M be a left $\mathcal{A}$-module. A *left connection on M* is a map $\nabla : \mathfrak{g} \times M \to M$ such that

$$
\begin{aligned}
&\nabla_\partial\big(m + m'\big) = \nabla_\partial m + \nabla_\partial m' \\
&\nabla_{\partial+\partial'} m = \nabla_\partial m + \nabla_{\partial'} m \\
&\nabla_{z\cdot\partial} m = z \nabla_\partial m \\
&\nabla_\partial(am) = a\nabla_\partial m + (\partial a)m
\end{aligned}
$$

for $m, m' \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$ and $z \in Z(\mathcal{A})$.

The curvature of ∇ is the map $R : \mathfrak{g} \times \mathfrak{g} \times M \to M$ defined as

$$R(\partial, \partial')m = \nabla_\partial \nabla_{\partial'} m - \nabla_{\partial'} \nabla_\partial m - \nabla_{[\partial,\partial']} m.$$

Connections on projective modules

Let $\mathcal{A}^n$ be a free (left) module and let $p \in \mathrm{End}_{\mathcal{A}}(\mathcal{A}^n)$ be a projection; i.e. $p^2 = p$, and let $P = p(\mathcal{A}^n)$ be the corresponding projective module.

Proposition

If $\widetilde{\nabla}$ is a connection on $\mathcal{A}^n$ then $\nabla = p \circ \widetilde{\nabla}$ is a connection on $P = p(\mathcal{A}^n)$.

Thus, connections always exist on projective modules; e.g. let $\{e_i\}_{i=1}^n$ be a basis of $\mathcal{A}^n$ and set (summation convention: sum i from 1 to n)

$$\widetilde{\nabla}_\partial(m^i e_i) = (\partial m^i)e_i.$$

Then $\widetilde{\nabla}$ is a connection on $\mathcal{A}^n$, and $p \circ \widetilde{\nabla}$ is a connection on $p(\mathcal{A}^n)$. Since every finitely generated projective module can be realized in this way, this shows that there exist connections on finitely generated projective modules.