Algebraic aspects of Riemannian geometry

Joakim Arnlind

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Connections and curvature

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Der}(\mathcal{A})$.

Definition

Let M be a left $\mathcal{A}$-module. A *left connection on* M is a map $\nabla : \mathfrak{g} \times M \to M$ such that

$$\begin{aligned}
\nabla_\partial\big(m + m'\big) &= \nabla_\partial m + \nabla_\partial m' \\
\nabla_{\partial+\partial'} m &= \nabla_\partial m + \nabla_{\partial'} m \\
\nabla_{z\cdot\partial} m &= z\nabla_\partial m \\
\nabla_\partial(am) &= a\nabla_\partial m + (\partial a)m
\end{aligned}$$

for $m, m' \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$ and $z \in Z(\mathcal{A})$.

The curvature of ∇ is the map $R : \mathfrak{g} \times \mathfrak{g} \times M \to M$ defined as

$$R(\partial, \partial')m = \nabla_\partial \nabla_{\partial'} m - \nabla_{\partial'} \nabla_\partial m - \nabla_{[\partial,\partial']} m.$$

Connections on projective modules

Let $\mathcal{A}^n$ be a free (left) module and let $p \in \mathrm{End}_{\mathcal{A}}(\mathcal{A}^n)$ be a projection; i.e. $p^2 = p$, and let $P = p(\mathcal{A}^n)$ be the corresponding projective module.

Proposition

If $\widetilde{\nabla}$ is a connection on $\mathcal{A}^n$ then $\nabla = p \circ \widetilde{\nabla}$ is a connection on $P = p(\mathcal{A}^n)$.

Thus, connections always exist on projective modules; e.g. let $\{e_i\}_{i=1}^n$ be a basis of $\mathcal{A}^n$ and set (summation convention: sum i from 1 to n)

$$\widetilde{\nabla}_\partial(m^i e_i) = (\partial m^i)e_i.$$

Then $\widetilde{\nabla}$ is a connection on $\mathcal{A}^n$, and $p \circ \widetilde{\nabla}$ is a connection on $p(\mathcal{A}^n)$. Since every finitely generated projective module can be realized in this way, this shows that there exist connections on finitely generated projective modules.

Metrics on vector bundles

Let $\mathcal{A}$ be the algebra of smooth functions on a differentiable manifold, let E be a vector bundle and let M denote the $\mathcal{A}$-module of smooth sections of E. A metric on the vector bundle E is a nondegenerate symmetric $\mathcal{A}$-bilinear map $g : M \times M \to \mathcal{A}$; i.e.

$$
\begin{aligned}
&g(m_1, m_2) = g(m_2, m_1) \qquad g(fm_1, m_2) = fg(m_1, m_2) \\
&g(m_1 + m_2, m_3) = g(m_1, m_3) + g(m_2, m_3) \\
&g(m_1, \cdot) : M \to M^* \quad \text{is bijective (for } m_1 \neq 0)
\end{aligned}
$$

for $m_1, m_2, m_3 \in M$ and $f \in \mathcal{A}$. Note: I haven't imposed "positive definite". This is a pseudo-Riemannian metric.

Hermitian forms on modules

Generalizing metrics to noncommutative algebras, one has to replace the symmetry condition $g(m_1, m_2) = g(m_2, m_1)$, which is far too restrictive for a noncommutative algebra.

Definition

Let M be a left $\mathcal{A}$-module. A map $h : M \times M \to \mathcal{A}$ is called a hermitian form on M if

$$
\begin{aligned}
h(m_1 + m_2, m_3) &= h(m_1, m_3) + h(m_2, m_3) \\
h(am_1, m_2) &= ah(m_1, m_2) \\
h(m_1, m_2)^* &= h(m_2, m_1).
\end{aligned}
$$

Moreover, we say that a hermitian form h is invertible if $h(m, \cdot) : M \to M^*$ is bijective for all $m \in M$ $(m \neq 0)$.

Thus, the (pseudo-)Riemannian metric is replaced by a (invertible) hermitian form in noncommutative geometry.

Hermitian forms on free and projective modules

Let $\mathcal{A}^n$ be a free (left) $\mathcal{A}$-module with basis $\{e_i\}_{i=1}^n$. A hermitian form h is defined as

$$h(m, n) = h(m^i e_i, n^j e_j) = m^i h_{ij}(n^j)^*$$

(summation from 1 to n implied) arbitrary $h_{ij} \in \mathcal{A}$ such that $h_{ij}^* = h_{ji}$.

For a free module, invertibility is equivalent to the existence of $h^{ij} \in \mathcal{A}$ s. t.

$$h^{ij} h_{jk} = \delta^i_k \mathbb{1}.$$

Thus, it is easy to construct hermitian forms on free modules. By restricting to the image of a projector $p : \mathcal{A}^n \to \mathcal{A}^n$ ($p^2 = p$), one can construct hermitian forms on projective modules.

Proposition (A. 2021)

Let h be a hermitian form on a finitely generated (left) projective $\mathcal{A}$-module M with generators $\{e_i\}_{i=1}^n$ and set $h_{ij} = h(e_i, e_j)$. Then h is invertible if and only if there exists $h^{ij} \in \mathcal{A}$ such that $h_{ij} h^{jk} e_k = e_i$ for $i = 1, \ldots, n$.

Metric connections

In Riemannian geometry, a connection is compatible with the metric if

$$X\big(g(m_1, m_2)\big) = g(\nabla_X m_1, m_2) + g(m_1, \nabla_X m_2)$$

for $m_1, m_2 \in M$ and X a vector field. Similarly, one defines a connection on a left $\mathcal{A}$-module to be compatible with a hermitian form h if

$$\partial h(m_1, m_2) = h\big(\nabla_\partial m_1, m_2\big) + h\big(m_1, \nabla_{\partial^*} m_2\big)$$

where $\partial^*(a) = \big(\partial(a^*)\big)^*$.

Theorem (A. 2021)

Let M be a finitely generated projective $\mathcal{A}$-module and let h be an invertible hermitian form on M. Then there exists a connection on M compatible with h.

Torsion of a connection

For a connection on the tangent bundle of a manifold, the torsion is defined as

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for $X, Y \in \mathcal{X}$ (vector fields). On a vector bundle there is no canonical way of defining torsion, but one can do it relative to an anchor map $\varphi : \mathcal{X} \to M$

$$T_\varphi(X, Y) = \nabla_X \varphi(Y) - \nabla_Y \varphi(X) - \varphi([X, Y]).$$

In noncommutative geometry, one can introduce

$$T_\varphi(\partial_1, \partial_2) = \nabla_{\partial_1} \varphi(\partial_2) - \nabla_{\partial_2} \varphi(\partial_1) - \varphi([\partial_1, \partial_2])$$

for $\partial_1, \partial_2 \in \mathrm{Der}(\mathcal{A})$ and $\varphi : \mathrm{Der}(\mathcal{A}) \to M$.

Noncommutative Levi-Civita connections

Given

- $*$-algebra $\mathcal{A}$
- Lie algebra of derivations $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$
- $\mathcal{A}$-module M
- invertible hermitian form h on M
- anchor map $\varphi : \mathfrak{g} \to M$

one can ask if there exists a torsion free connection on M compatible with h; i.e. a connection ∇ om M such that

$$\partial h(m_1, m_2) = h\big(\nabla_\partial m_1, m_2\big) + h\big(m_1, \nabla_{\partial^*} m_2\big)$$
$$T_\varphi(\partial_1, \partial_2) = \nabla_{\partial_1}\varphi(\partial_2) - \nabla_{\partial_2}\varphi(\partial_1) - \varphi([\partial_1, \partial_2]) = 0$$

for $\partial, \partial_1, \partial_2 \in \mathfrak{g}$ and $m_1, m_2 \in M$? Furthermore, if such a "Levi-Civita connection" exists, is it unique?

Pseudo-Riemannian calculi

Without further assumptions, neither existence nor uniqueness of a Levi-Civita connection is guaranteed. Over the last 10 years I've been interested in understanding under what assumptions such connections exist, and when they are unique.

Let $M_\varphi \subseteq M$ denote the image of φ in M. Assume that

- M_φ generates M as an $\mathcal{A}$-module,
- $h(E, E') = h(E, E')^*$ for $E, E' \in M_\varphi$
- $h\big(E, \nabla_\partial E'\big)^* = h\big(E, \nabla_\partial E'\big)$ for $E, E' \in M_\varphi$ and $\partial \in \mathfrak{g}$.

Theorem (A., Wilson 2017)

Under the above assumptions, there exist at most one Levi-Civita connection on M.

It is easy to show that Levi-Civita connection exist on free modules. However, what about projective? Not so clear for the moment.