Graded Ring Theory and Leavitt Path Algebras

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Conventions

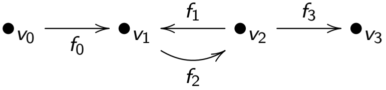
Definition

A ring R is a set equipped with two binary operations: addition and multiplication. $(R, +)$ is an abelian group and $(R, *)$ is a semigroup. Moreover, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in R$.

1. Not necessarily equipped with a multiplicative identity element! Also called: Rng.
2. Always associative
3. Often: "local" multiplicative identity elements. Eg. s-unital rings.

Path algebras I

Directed graph:



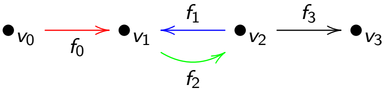
Vertices: v_0, v_1, v_2, v_3
Edges: f_0, f_1, f_2, f_3.
Start and range: e.g. $s(f_0) = v_0, r(f_0) = v_1$.

Definition

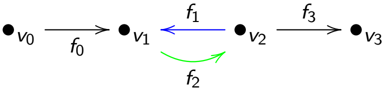
A directed graph $E = (E^0, E^1, s, r)$ is two sets E^0, E^1 and two maps $s, r \colon E^1 \to E^0$.

Path algebras II

Paths: sequence of edges (left to right)
$\alpha = f_0 f_2 f_1$.



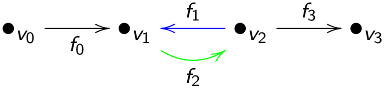
$\beta = f_1 f_2$.



Path algebras III

Multiplication = path concatenation

Consider
$\gamma = f_1, \delta = f_2$.



$$\gamma\delta = f_1 f_2 \neq f_2 f_1 = \delta\gamma$$

Algebraic analogues of C^*-algebras

There is a trend to obtain algebraic analogues of C^*-algebras (roots in the works of von Neumann and Kaplansky)

C^*-algebra	Algebra
Cuntz C^*-algebra	Leavitt algebra
Graph C^*-algebra	Leavitt path algebra
Cuntz-Pimsner C^*-algebra	Algebraic Cuntz-Pimsner ring
Groupoid C^*-algebra	Steinberg algebra
Crossed products by partial actions	Unital partial crossed products

1. Leavitt path algebras (introduced by Ara, Moreno and Pardo 2004 and by Abrams and Aranda Pino 2005)
2. algebraic Cuntz-Pimsner rings (introduced by Carlsen and Ortega, 2008)

Epsilon-strongly graded rings

Some of these analogues are non-commutative group graded algebras!
Special classes of group graded rings:

1. strongly graded rings (studied by Dade et. al 1980s)
2. epsilon-strongly graded rings (introduced by Nystedt, Öinert and Pinedo 2016)
3. nearly epsilon-strongly graded rings (introduced by Nystedt and Öinert 2018)

Conventions

1. A ring is associative but not necessary equipped with a multiplicative identity element.
2. A ring with a multiplicative identity element $1 \neq 0$ is called *unital*. Note that the trivial ring is not unital!
3. $\mathbb{Z}$ will be used to denote the infinite cyclic group.

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Example 1

Example

Consider the ring of compactly supported real-valued continous functions with point-wise multiplication. No multiplicative identity element!

Definition

A ring R is called *s-unital* if $x \in xR \cap Rx$ for every $x \in R$.

Remark

If R is s-unital, then $R^2 = R$. In other words, R is an idempotent ring.

Example

$R = 2\mathbb{Z}$ is a ring which does not admit a multiplicative identity element. $R^2 = 4\mathbb{Z} \subsetneq R$. Thus R is not s-unital.

Locally unital rings

Proposition

The ring R is s-unital if and only if for all $n \in \mathbb{N}$ and all $r_1, r_2, \ldots, r_n \in R$ there is some $e \in R$ such that $er_i = r_i = r_i e$ for all $i \in \{1, 2, \ldots, n\}$.

Definition

We say that R is *locally unital* if for all $n \in \mathbb{N}$ and all $r_1, r_2, \ldots, r_n \in R$ there is some idempotent $e \in R$ such that $er_i = r_i = r_i e$ for all $i \in \{1, \ldots, n\}$.

Example

The previous example is s-unital but NOT locally unital.

von Neumann regular rings are locally unital

Definition

A ring R is called *von Neumann regular* if for every $a \in R$ there exists some $x \in R$ such that $a = axa$.

Example

A field is regular, $\mathbb{Z}$ is not regular.

Proposition

(Ánh, Márki, 1987) Every von Neumann regular ring is locally unital.

Sets of local units

Definition

1. Let $e, e' \in R$. Put $e \vee e' := e + e' - ee'$.
2. If e, f are idempotents of R, then they are called *orthogonal* if $ef = fe = 0$.

Definition

(Abrams, 1983) Let E be a set of commuting idempotents of R which is closed under the $\vee$-operator. Then E is called *a set of local units* for R if for all $r \in R$ there is some $e \in E$ such that $er = r = re$.

Proposition

If a ring R admits a set of local units, then R is locally unital.

Enough idempotents

Definition

(Fuller, 1976) The ring R is said to have *enough idempotents* in case there exists a set $\{e_i\}_{i \in I}$ of orthogonal idempotents such that $R = \bigoplus_{i \in I} Re_i = \bigoplus_{i \in I} e_i R$. The set $\{e_i\}$ is called a *complete set of idempotents*.

Example

Let R be a unital ring. Consider the external direct sum $S = \bigoplus_{\mathbb{Z}} R$. S becomes a non-unital ring with component-wise addition and multiplication. Fix $i \in \mathbb{Z}$. For $j \in \mathbb{Z}$, put

$$e_i(j) := \begin{cases} 1_R & \text{if } i = j \\ 0_R & \text{otherwise} \end{cases}.$$

The set $\{e_i\}_{i \in \mathbb{Z}}$ is a complete set of idempotents for S.

Example and Summary

Example

Let R be a unital ring. Consider generalized $\mathbb{N} \times \mathbb{N}$ matrices with only finitely many non-zero entries. The generalized matrix ring $M_{\mathbb{N}}$ has enough idempotents but is not unital.

Summary:

$$\{ \text{ unital rings } \} \subsetneq \{ \text{ rings with enough idempotents } \} \subsetneq \{ \text{ locally unital rings } \} \subsetneq$$
$$\subsetneq \{ \text{ s-untial rings } \} \subsetneq \{ \text{ rings } \}$$

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Leavitt path algebras

1. Introduced by Ara, Moreno and Pardo 2004 and by Abrams and Aranda Pino 2005. Algebraic analogue of graph C^*-algebras.
2. Let R be a (possibly non-commutative) unital ring and let E be a directed graph. Associate an R-algebra $L_R(E)$ with the graph E.
3. Many rings are realizable as Leavitt path algebras
4. Focus has been on coefficients in a field.

Conjecture (Abrams-Tomforde)

Let E, F be directed graphs. If $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ as rings, then $C^*(E) \cong C^*(F)$ as C^*-algebras.

Leavitt path algebra definition

Definition

Let R be a ring and $E = (E^0, E^1, s, r)$ be a directed graph. The *Leavitt path algebra* attached to E with coefficients in R is the free associative R-algebra generated by the symbols:

1. $\{v \mid v \in E^0\}$,
2. $\{f \mid f \in E^1\}$,
3. $\{f^* \mid f \in E^1\}$.

...

Leavitt path algebra: Definition

Definition

...

subject to the following relations:

a. $v_i v_j = \delta_{i,j} v_i$ for all $v_i, v_j \in E^0$,
b. $s(f)f = fr(f) = f$ and $r(f)f^* = f^*s(f) = f^*$ for all $f \in E^1$,
c. $f^*f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$,
d. $\sum_{f \in E^1, s(f)=v} ff^* = v$ for all $v \in E^0$ for which $s^{-1}(v)$ is non-empty and finite.

We let R commute with the generators.

Leavitt path algebras: Remarks

1. A *path* is a sequence $\alpha = f_1 f_2 \dots f_n$ such that $s(f_{i+1}) = r(f_i)$ for $i \in \{1, \dots, n-1\}$.
2. A *ghost path* is a sequence $\beta^* = f_n^* f_{n-1}^* \dots f_1^*$. 'backwards-path'.

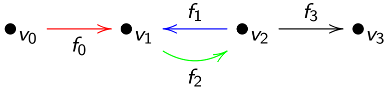
Proposition

Let R be a unital ring and let E be a directed graph. Then

$$L_R(E) = Span_R\{\alpha\beta^* \mid \alpha, \beta \in Path(E), r(\alpha) = r(\beta)\}.$$

Leavitt path algebras: Examples I

Ex: Consider the LPA associated with

E :


Elements in $L_R(E)$:
$\alpha^* = f_1^* f_2^* f_0^* \in L_R(E)$
$v_0 \in L_R(E)$
$\gamma = f_0 \in L_R(E)$

$\alpha^* \gamma = f_1^* f_2^* f_0^* f_0 = f_1^* f_2^* r(f_0) = f_1^* f_2^*.$

Leavitt path algebras: Examples II

Example

$$A_1 : \qquad \bullet_v$$

In this case, $L_R(A_1) \cong Rv \cong R$.

Example

E_1 :

In this case, $L_R(E_1) \cong_\phi R[x, x^{-1}]$ via the map defined by $\phi(v) = 1_R, \phi(f) = x, \phi(f^*) = x^{-1}$.

Leavitt path algebras: Examples III

Example

$$A_2 : \qquad \bullet_{v_1} \xrightarrow{f} \bullet_{v_2}$$

In this case, $L_R(A_2) \cong M_2(R)$.

Generalization:

Example

$$A_n : \qquad \bullet_{v_1} \longrightarrow \bullet_{v_2} \cdots\cdots \longrightarrow \bullet_{v_{n-1}} \longrightarrow \bullet_{v_n}$$

In this case, $L_R(A_n) \cong M_n(R)$.

Leavitt path algebras: Examples IV

The previous graphs have all been finite, but we also allow infinite graphs!

Example

Infinitely many vertices:

$$E' : \bullet_{v_1} \quad \bullet_{v_2} \quad \bullet_{v_3} \quad \bullet_{v_4} \quad \bullet_{v_5} \quad \bullet_{v_6} \quad \bullet_{v_7} \quad \bullet_{v_8} \quad \bullet_{v_9} \quad \bullet_{v_{10}} \cdots$$

In this case, $L_R(E') \cong \bigoplus_{i>0} Rv_i$.

Example

$$E'' : \bullet_{v_1} \xrightarrow{(\infty)} \bullet_{v_2}$$

Examples V

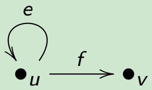
Example

Take $n \geq 2$. Let R_n denote the rose with n petals graph having one vertex and n loops. Then, $L_K(R_n) \cong L_K(1, n)$ where $L_K(1, n)$ is the Leavitt algebra of type $(1, n)$.

Example

Consider the Toeplitz graph:

E_T :



Then $L_K(E)$ is isomorphic to the algebraic Toeplitz K-algebra investigated by Jacobson in 1950.

Examples VI

Direct limits of graphs:

Example

$$A_{\mathbb{N}}: \qquad \bullet_{v_1} \longrightarrow \bullet_{v_2} \cdots\cdots \longrightarrow \bullet_{v_{n-1}} \longrightarrow \cdots$$

In this case, $L_R(A_{\mathbb{N}}) \cong \lim_{n\in\mathbb{N}} M_n(R) = M_{\mathbb{N}}(R)$.

Examples VII: A conjecture by Kaplansky

Conjecture

(Kaplansky, 1970) Is a regular prime ring necessarily primitive? It seems unlikely that the answer is affirmative, but a counter-example may have to be weird.

A first example was given in 1977 by Domanov. The theory of LPAs allows for a infinite class of examples to be constructed.

Definition

(Abrams, Bell, Rangaswamy, 2014) Let X be a set and let $E_F(X)$ denote all finite subsets of X. The graph $E_F(X)$ is defined as $E_F(X)^0 = E_F(X)$ and $E_F(X)^1 = \{(A, A') \mid A \subsetneqq A'\}$.

Proposition

(Abrams, Bell, Rangaswamy, 2014) Let X be any uncountable set and let K be any field. The Leavitt path algebra $L_K(E_F(X))$ is prime, non-primitive and von Neumann regular.

Unital Leavitt path algebras

Proposition ([1, Lem. 1.2.12)

Let R be a unital ring and let E be a directed graph. Consider the Leavitt path algebra $L_R(E)$. Then $L_R(E)$ is a unital ring if and only if E has finitely many vertices. In this case,

$$1_{L_R(E)} = \sum_{v \in E^0} v.$$

Example

The ring $L_R(E')$ is not unital.

Proposition

The Leavitt path algebra $L_K(E)$ is a ring with enough idempotens.

The $\mathbb{Z}$-graded structure of Leavitt path algebras

Definition

The canonical $\mathbb{Z}$-grading of $L_R(E)$ is defined by,

$$\deg(\alpha\beta^*) = \mathsf{len}(\alpha) - \mathsf{len}(\beta).$$

Hazrat [**hazrat2013graded**] gave a criteria for the grading to be strong. Later, extended by Nystedt and Öinert [**nystedt2017epsilon**].

$$\begin{array}{ccccc} E \text{ finite with no sinks} & \Rightarrow & E \text{ finite} & \Rightarrow & E \text{ a graph} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ L_R(E) \text{ unital strong} & \Rightarrow & L_R(E)\ \epsilon\text{-strong} & \Rightarrow & L_R(E) \text{ nearly } \epsilon\text{-strong} \end{array}$$

Figure: The graded structure of Leavitt path algebras.

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Group graded rings

Definition

Let G be a group and let S be a ring. A *grading* of S is a collection of additive subsets of S, $\{S_g\}_{g\in G}$, such that

$$S = \bigoplus_{g\in G} S_g,$$

and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. The ring S is called a *G-graded ring*. S_e is called the *principal component*.

Proposition

Let S be a G-graded ring. Then S_e is a subring of S.

Remark

A given ring can in general be equipped with numerous different gradings.

Graded rings: Examples I

Example

The Laurent polynomial ring is $\mathbb{Z}$-graded by,

$$R[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rx^i.$$

Principal component: $(R[x, x^{-1}])_0 = R$.

Example

(The group ring) Let G be a group and let R be a unital ring. The group ring $R[G] = \bigoplus_{g \in G} R\delta_g$ is naturally G-graded.

Strongly graded rings

Definition

(Dade, 1980) A G-grading $\{S_g\}$ of a ring S is called *strong* if $S_g S_h = S_{gh}$ holds for all $g, h \in G$. The ring S is called *strongly* G-graded.

Example

Let R be a unital ring. Then $(Rx^i)(Rx^j) = Rx^{i+j}$ for all $i, j \in \mathbb{Z}$. Hence, the Laurent polynomial ring $R[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rx^i$ is strongly $\mathbb{Z}$-graded.

Example

Let G be a group and let R be a unital ring. The group ring $R[G]$ is strongly G-graded.

Strongly graded rings: examples

Example

(Skew group ring) Let R be a unital ring, let G be a group and let $\phi\colon G \to \mathrm{Aut}(R)$ be a group homomorphism. The skew group ring $R \star_\phi G$ has the same additional structure as $R[G]$. Multiplication is defined by $(a\delta_g)(b\delta_h) = a\phi(g)(b)\delta_{gh}$.

Example

(Twisted group ring) Let R be a unital ring, let G be a group and let $\alpha\colon G \times G \to \mathrm{U}(R)$ be a map satisfying (i) $\alpha(g,h)\alpha(gh,s) = \alpha(h,s)\alpha(g,hs)$ and (ii) $\alpha(g,e) = \alpha(e,g) = 1_R$ for all $g,h,s \in G$. The twisted group ring $R \star^\alpha G$ has the same additional structure as $R[G]$. Multiplication is defined by $(a\delta_g)(b\delta_h) = ab\alpha(g,h)\delta_{g,h}$.

skew + twist = algebraic crossed product. Studied by Dade, Passman, et al ca. 1980s.

Strongly graded rings: examples

Definition

A *crossed system* is a quadruple (R, G, σ, α), where R is a unital ring, G a group, $\sigma\colon G \to \mathrm{Aut}(R)$ a group homomorphism, $\alpha\colon G \times G \to U(R)$ a map satisfying the following conditions:

1. $\sigma_g(\sigma_h(a)) = \alpha(g,h)\sigma_{gh}(a)\alpha(g,h)^{-1}$
2. $\alpha(g,h)\alpha(gh,s) = \sigma_g(\alpha(h,s))\alpha(g,hs)$
3. $\alpha(g,e) = \alpha(e,g) = 1_R$

for all $g, h, s \in G$ and $a \in R$.

Definition

Given a crossed system (R, G, σ, α) we define the *algebraic crossed product* $R \star_\sigma^\alpha G$ with the same additive structure as $R[G]$ but with multiplication defined by

$$(a\delta_g)(b\delta_h) = a\sigma_g(b)\alpha(g,h)\delta_{gh}.$$

Strongly graded rings: characterizations

Remark

Dade gave a categorical characterization of strongly graded ring (Dade's theorem). Categorical connection between the principal component S_e and S.

Lemma

Let S be a G-graded ring. Then $S_g S_{g^{-1}}$ is an ideal of S_e.

Proposition

A unital G-graded ring $S = \bigoplus_{g \in G} S_g$ is strongly G-graded if and only if $1_S \in S_g S_{g^{-1}}$ for every $g \in G$.

Dade's Theorem (I)

Definition

Let S be a unital G-graded ring. Let M be a unital left S-module and let $\{M_x\}_{x\in G}$ be a family of additive subsets of M satisfying $M = \bigoplus_{g\in G} M_x$ and $S_g M_x \subseteq M_{gx}$. Then the pair $(M, \{M_x\})$ is called a *graded left S-module*.

Let $S - \text{gr}$ denote the category of graded left S-modules.

Example

Let K be a field and consider the polynomial ring $S := K[x]$. Recall that S is $\mathbb{Z}$-graded by putting $S_n = Kx^n$ for $n \geq 0$ and $S_n := \{0\}$ for $n < 0$. Consider the ideal I generated by x. I is a graded left S-module.

Dade's Theorem (II)

A functor $\mathrm{Ind}\colon S_e - \mathrm{mod} \to S - \mathrm{gr}$ is defined as follows. For $N \in S_e - \mathrm{mod}$ and $N_1, N_2 \in S_e - \mathrm{mod}$, $f \in \mathrm{Hom}_{S_e}(N_1, N_2)$

$$\mathrm{Ind}(N) := (S \otimes_{S_e} N, \{S_g \otimes_{S_e} N\}_{g \in N})$$
$$\mathrm{Ind}(f) := \mathrm{id}_S \otimes_{S_e} f$$

Theorem

(Dade, 1980) Let S be a unital G-graded ring. Then S is strongly G-graded if and only if Ind is a equivalence of categories.

Remark

If S is unital strongly graded then $S - \mathrm{gr} \simeq S_e - \mathrm{mod}$. However, the converse is not true.

Why epsilon-strongly graded rings?

Introduced by Nystedt, Öinert and Pinedo.

1. The class of strongly graded rings too small to include all our examples.
2. Natural generalization of unital strongly graded rings.
3. Some results for strongly graded rings seems to generalize to epsilon-strongly graded rings!

Epsilon-strongly graded rings: Definition

Definition

Let R be a ring. An ideal I of R is called *unital* if I has a multiplicative identity element $e \in I$ such that $ex = xe = x$ for every $x \in I$.

Definition (Nystedt, Öinert, Pinedo, 2016)

Let S be a G-graded ring. If $S_g S_{g^{-1}}$ is a unital ideal of S_e for every $g \in G$,

$$S_g S_h = S_g S_{g^{-1}} S_{gh} \qquad \forall g, h \in G \tag{1}$$

and,

$$S_g S_h = S_{gh} S_{h^{-1}} S_h \qquad \forall g, h \in G. \tag{2}$$

Then S is called *epsilon-strongly G-graded*.

Epsilon-strongly graded rings: Remarks

Remark

unital strongly graded $\implies$ epsilon-strongly graded.

Remark

The multiplicative identity element of $S_g S_{g^{-1}}$ is denoted by ϵ_g. Note that we might have $\epsilon_g = 0$ for some $g \in G$.

Theorem (Nystedt, Öinert, Pinedo)

Let S be an epsilon-strongly G-graded ring. Then S is strongly G-graded if and only if $\epsilon_g = 1_S$ for every $g \in G$.

Epsilon-strongly graded rings: Example

Example

Consider the $\mathbb{Z}$-grading of the ring $M_2(\mathbb{C})$ given by,

$$(M_2(\mathbb{C}))_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix},$$

$$(M_2(\mathbb{C}))_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix},$$

$$(M_2(\mathbb{C}))_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix},$$

and $(M_2(\mathbb{C}))_i = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right\}$ for $|i| > 1$.

Example cont.

Example

Note that,

$$(M_2(\mathbb{C}))_1(M_2(\mathbb{C}))_{-1} = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix}, (M_2(\mathbb{C}))_{-1}(M_2(\mathbb{C}))_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{pmatrix},$$

are unital ideals of $(M_2(\mathbb{C}))_0$ with multiplicative identity elements:

$$\epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, a routine check shows that the relations hold.
Thus, $M_2(\mathbb{C})$ is epsilon-strongly $\mathbb{Z}$-graded.

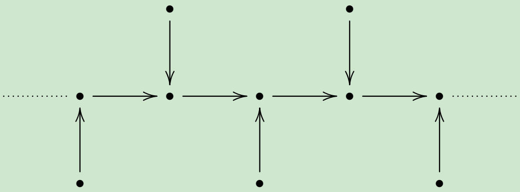
Epsilon-strongly graded rings: Properties

Proposition (Lännström, 2018)

If S is a non-trivial epsilon-strongly G-graded ring, then S is unital.

Example ([hazrat2013graded])

Let R be a unital ring and let E be the following directed graph.



Then $L_R(E)$ is strongly $\mathbb{Z}$-graded but not unital.

Examples

Examples of epsilon-strongly graded rings:

1. unital strongly graded rings (studied by Dade, Passman, et al during 1980s)
2. unital partial crossed products (Nystedt, Öinert, Pinedo, 2016 [**nystedt2016epsilon**]),
3. Leavitt path algebras of finite graphs (Nystedt, Öinert, 2017),
4. corner skew Laurent polynomial rings (Lännström, 2019).

Epsilon-strongly graded rings: Equivalent characterizations

Proposition

(Nystedt, Öinert, Pinedo 2018) Let S be a G-graded ring. If, for every $g \in G$ and $s \in S_g$ there exist some $\epsilon_g \in S_g S_{g^{-1}}, \epsilon'_g \in S_{g^{-1}} S_g$ such that $\epsilon_g s = s = s\epsilon'_g$, then S is epsilon-strongly G-graded.

Definition

Let S be a G-graded ring. If $S_g = S_g S_{g^{-1}} S_g$ for every $g \in G$, then S is called *symmetrically* G-graded.

Proposition

(Nystedt, Öinert, Pinedo 2018) Let S be a G-graded ring. Then S is epsilon-strongly G-graded if and only if S is symmetrically G-graded and $S_g S_{g^{-1}}$ is an unital ideal for every $g \in G$.

Nearly epsilon-strongly graded rings

Definition (Nystedt, Öinert, 2019)

Let S be a G-graded ring. If, for every $g \in G$ and $s \in S_g$ there exist some $\epsilon_g(s) \in S_g S_{g^{-1}}, \epsilon_g(s)' \in S_{g^{-1}} S_g$ such that $\epsilon_g(s)s = s = s\epsilon_g(s)'$, then S is called *nearly epsilon-strongly G-graded*.

Remark

Let E be a infinite graph. Then the canonical $\mathbb{Z}$-grading of $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded but not epsilon-strongly $\mathbb{Z}$-graded.

Proposition

(Nystedt, Öinert, 2019) Let S be a G-graded ring. Then S is nearly epsilon-strongly G-graded if and only if S is symmetrically G-graded and $S_g S_{g^{-1}}$ is a s-unital ideal for every $g \in G$.

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Induced quotient group grading: Example

Example

Let R be a ring and consider the Laurent polynomial ring $R[x, x^{-1}]$ with the standard $\mathbb{Z}$-grading. That is,

$$R[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rx^i. \tag{3}$$

Consider the quotient group $\mathbb{Z}/2\mathbb{Z}$ of $\mathbb{Z}$. There is a naturally induced $\mathbb{Z}/2\mathbb{Z}$-grading of $R[x, x^{-1}]$ given by:

$$R[x, x^{-1}] = \left(\bigoplus_{i \in 2\mathbb{Z}} Rx^i \right) \oplus \left(\bigoplus_{i \in 1+2\mathbb{Z}} Rx^i \right) = S_{[0]} \oplus S_{[1]},$$

where $[0]$ denotes the class $0 + 2\mathbb{Z}$ and $[1]$ denotes the class $1 + 2\mathbb{Z}$.

Induced quotient group grading: Definition

Definition

Let G be a group and let N be a normal subgroup of G. Let $S = \bigoplus_{g \in G}$ be a G-graded ring. For every class $C \in G/N$, put

$$S_C := \bigoplus_{g \in C} S_g.$$

Lemma

$\{S_C\}_{C \in G/N}$ is a G/N-grading of S.

The G/N-grading $\{S_C\}_{C \in G/N}$ is called the *induced quotient group grading* of S.
If $\{S_g\}_{g \in G}$ is a strong G-grading, then $\{S_C\}_{C \in G/N}$ is strong!

Induced gradings of epsilon-strongly graded rings

Proposition (Nystedt, Öinert, Pinedo 2016)

The unital partial crossed products are epsilon-strongly graded. In particular, unital partial skew group rings are epsilon-strongly graded.

Proposition (Lännström, 2018)

There is a unital partial skew group ring by $\mathbb{Z}$ such that the induced $\mathbb{Z}/2\mathbb{Z}$-grading is not epsilon-strong.

Theorem (Lännström, 2018)

Let S be an epsilon-strongly G-graded ring. If S_e is left (right) noetherian, then, for any normal subgroup of N, the induced G/N-grading is epsilon-strong.

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Research: A Hilbert Basis Theorem

Definition

A unital ring R is called left (right) noetherian if for any ascending chain of left (right) ideals,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots,$$

there exists some integer $k > 0$ such that $I_k = I_{k+1} = I_{k+2} = I_{k+3} = \dots$.

Idea 1

("Hilbert Basis Theorem") Under certain conditions on the grading, the ring S is noetherian if and only if S_e is noetherian.

Hilbert Basis Theorem for epsilon-strongly graded rings

Theorem

*(Lännström, 2018, cf. Bell (1987) [***bell1987localization***]) Let G be a polycyclic-by-finite group and let S be an epsilon-strongly G-graded ring. Then, S is left/right noetherian if and only if S_e is left/right noetherian.*

Remark

1. Bell (1987) [**bell1987localization**] proved the analogous statement for strongly graded rings.
2. The proof relies on the theorem from Paper A.

Artinian epsilon-strongly graded rings

Theorem

(Lännström, 2018) Let G be a torsion-free group and let S be an epsilon-strongly G-graded ring. Then, S is left/right artinian if and only if S_e is left/right artinian and $S_g \neq \{0\}$ for finitely many $g \in G$.

Applications

Application 1

Characterizations of noetherian and artinian unital partial crossed products. Generalizes previous work on partial skew group rings by Carvalho, Cortes, Ferrero.

Application 2

Application to Leavitt path algebras.
$\mathbb{Z}$ is both polycyclic-by-finite and torsion-free. Hence, we can apply the above theorems to the special case of Leavitt path algebras of finite graphs.

Characterization of noetherian and artinian Leavitt path algebras

For coefficients in a field: (Abrams, Ara, Siles Molina)
For coefficients in a commutative ring: (Steinberg 2018 [**steinberg2018chain**])

Theorem (Lännström, 2018)

Let R be a ring and E a directed graph. Then the following assertions hold.

1. *$L_R(E)$ is left (right) noetherian if and only if R is left (right) noetherian and E is a finite graph containing no cycles with exits.*
2. *$L_R(E)$ is left (right) artinian if and only if R is left (right) artinian and E is a finite acyclic graph.*

Conditions on E + conditions on $R \iff$ conditions on $L_R(E)$

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Von Neumann regular rings

Definition (Von Neumann, 1936)

A unital ring R is called *von Neumann regular* if for each $a \in R$ there is some $x \in R$ such that $a = axa$.

Generalizes to non-unital rings in an obvious manner.

Definition

A ring R is called *s-unital* if $x \in Rx \cap xR$ for every $x \in R$.

Proposition

Let R be an s-unital ring. The following are equivalent:

1. R is *von Neumann regular*
2. every finitely generated left (right) ideal is generated by an idempotent

Graded von Neumann regular rings

Definition (Năstăsescu, Oystaeyen, 1982 [nastasescu1982graded])

Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. If, for each $g \in G$ and every $a \in S_g$, there is some $x \in S$ such that $a = axa$, then S is called *graded von Neumann regular*.

Proposition (Hazrat, 2014)

Let S be a G-graded ring with homogeneous local units. Then the following are equivalent:

1. S is graded von Neumann regular
2. Any finitely generated right (left) graded ideal of S is generated by a homogeneous idempotent.

Graded von Neumann regular rings and strongly graded rings

Definition

A G-grading $S = \bigoplus_{g \in G} S_g$ is called *strong* if $S_g S_h = S_{gh}$ for all $g, h \in G$.

Theorem (Năstăsescu, Oystaeyen, 1982 [nastasescu1982graded])

Let $S = \bigoplus_{g \in G} S_g$ be a unital strongly G-graded ring. Then S is graded von Neumann regular if and only if S_e is von Neumann regular.

Example

Let R be a von Neumann regular ring. Then the group ring $R[G]$ and the Laurent polynomial ring $R[x, x^{-1}]$ are graded von Neumann regular.

Necessary conditions for being graded von Neumann regular

Proposition

Let $S = \bigoplus_{g \in G} S_g$ be a graded von Neumann regular ring. Then S_e is von Neumann regular.

Proposition (Lännström, 2019)

Let $S = \bigoplus_{g \in G} S_g$ be a graded von Neumann regular ring. Then S is nearly epsilon-strongly G-graded.

Proof

Take $g \in G$ and $s \in S_g$. We need to find some $\epsilon_g(s) \in S_g S_{g^{-1}}$ and $\epsilon_g(s)' \in S_{g^{-1}} S_g$ such that $\epsilon_g(s)s = s = \epsilon_g(s)' s$.

Since S is graded von Neumann regular, there is some $b \in S_{g^{-1}}$ such that $s = sbs$. Take $\epsilon_g(s) := sb$ and $\epsilon_g(s)' = bs$.

A characterization of graded von Neumann regular rings

Necessary conditions are also sufficient! (cf. Yahya, 1997)

Theorem (Lännström, 2019)

Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Then S is graded von Neumann regular if and only if S is nearly epsilon-strongly G-graded and S_e is von Neumann regular.

Remark

Generalizes Năstăsescu, Oystaeyen theorem for unital strongly graded rings!

Leavitt path algebras over fields are graded von Neumann regular

Theorem (Hazrat, 2014 [hazrat2014leavitt])

Let K be a field and let E be a directed graph. Then $L_K(E)$ is graded von Neumann regular.

Example

Let R be a unital ring that is not von Neumann regular.

$$A_1 : \qquad \bullet_v$$

It can be shown that $L_R(E) \cong_{\mathrm{gr}} R$ is graded von Neumann regular if and only if R is von Neumann regular. Hence, $L_R(E)$ is not graded von Neumann regular.

Leavitt path algebras with coefficients in a general unital ring

Main goal of this project:

Theorem (Lännström, 2019)

Let R be a unital ring and let E be a directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

Proof idea:
R von Neumann regular $\rightsquigarrow$ $L_R(E)_0$ is von Neumann regular $\rightsquigarrow$ $L_R(E)$ graded von Neumann regular.

Proposition (Nystedt, Öinert, 2018)

Let R be a unital ring. If E is a directed graph, then $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded. If E is a finite directed graph, then $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded.

Example

1. $\mathbb{C}$ is von Neumann regular
2. $\mathbb{Z}$ is not von Neumann regular

By Theorem, for any graph E,

1. $L_{\mathbb{C}}(E)$ is graded von Neumann regular
2. $L_{\mathbb{Z}}(E)$ is not graded von Neumann regular

Proof of Theorem (I)

Lemma (Lännström, 2019)

Let R be a unital ring and let E be a finite directed graph. Then $(L_R(E))_0$ is von Neumann regular if and only if R is von Neumann regular.

Corollary (Lännström, 2019)

Let R be a unital ring and let E be a finite directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

Proof.

$L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded (Nystedt, Öinert, 2017).
$L_R(E)$ is graded von Neumann regular if and only if $(L_R(E))_0$ is von Neumann regular.
By Lemma, $(L_R(E))_0$ is von Neumann regular if and only if R is von Neumann regular. □

Proof of Theorem (II)

Technical reduction step: Let E be any graph. Then $L_R(E)$ is the direct limit of Leavitt path algebras of finite graphs.

Theorem (Lännström, 2019)

Let R be a unital ring and let E be any (possibly non-finite) directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

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Connell's Theorem

Definition

A ring R is called prime if for all ideals A, B of R, $AB = 0$ implies that $A = 0$ or $B = 0$.

Theorem (Connell, 1963)

Let R be a unital ring and let G be a group. The group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroups.

Passman's Theorem

Definition (Passman, 1984)

Let S be a unital strongly G-graded ring. Then G acts on the ideals of S_e by $I^x := S_{x^{-1}} I S_x$. Let H be a subgroup of G. If $I^x = I$ for every $x \in H$, then I is called H-invariant.

Theorem (Passman, 1984)

Let S be a unital strongly G-graded ring. Then S is not prime if and only if there exist:

1. subgroups $N \lhd H \subseteq G$ with N finite;
2. an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and
3. nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Invariant ideals

Updated definition:

Definition (L., Lundström, Wagner, Öinert, 2021, cf. Passmann)

Let S be a G-graded ring.

1. For a subset $I \subseteq S$ and $x \in G$, we consider the set $I^x := S_{x^{-1}} I S_x$.
2. Let H be a subgroup of G. We say that I is H-invariant if $I^x \subseteq I$ for every $x \in H$.
3. Let N be a normal subgroup of H. We say that I is H/N-invariant if $S_{C^{-1}} I S_C \subseteq I$ for every $C \in H/N$.

Not a group action! $(I^x)^y \neq I^{xy}$.

Our main result

Theorem (L., Lundström, Öinert, Wagner 2021)

Let S be a nearly epsilon-strongly G-graded ring. Then S is not prime if and only if there exist:

1. subgroups $N \lhd H \subseteq G$ with N finite;
2. an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and
3. nonzero H/N-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Remark

Let S be a unital strongly G-graded ring. "Passman"-invariant ideal coincide with invariant ideal ($I^x \subseteq I \iff I^x = I$). H/N-invariant implies H-invariant.

Torsion-free grading groups

Definition

A proper G-invariant ideal Q of S_e is called *G-prime* if for all G-invariant ideals A, B of S_e we have $A \subseteq Q$ or $B \subseteq Q$ whenever $AB \subseteq Q$. The ring S_e is called G-prime if $\{0\}$ is a G-prime ideal of S_e.

Theorem (L., Lundström, Öinert, Wagner, 2021)

Suppose that G is torsion-free and that S is nearly epsilon-strongly G-graded. Then S is prime if and only if S_e is G-prime.

Application to Prime Leavitt path algebras

Definition

Let E be a directed graph. The graph E is said to satisfy Condition (MT-3) if for every pair of vertices $u, v \in E^0$, there is some vertex $w \in E^0$ such that there are paths (possibly of zero length) from u to w and from v to w. (Confluence vertex w).

Example

$$A_n : \qquad \bullet_{v_1} \xrightarrow{f_1} \bullet_{v_2} \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \bullet v_n$$

Note that A_n satisfies (MT-3).

Application to Prime Leavitt path algebras

The following theorem generalizes work by Abrams-Bell-Rangaswamy and Larki:

Theorem (L., Lundström, Öinert, Wagner 2021)

Let $L_R(E)$ be a Leavitt path algebra over a unital ring R. Then $L_R(E)$ is prime if and only if R is prime and E satisfied Condition (MT-3).

Example

Take $E := A_n$ in the above theorem. Then $L_R(A_n) \cong M_n(R)$ is prime if and only if R is prime.

Unital partial crossed products

Definition

A *unital twisted partial action of G on R* is a triple $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G})$ where for each $g \in G$, the D_g's are unital ideals of R, $\alpha_g \colon D_{g^{-1}} \to D_g$ are ring isomorphisms and for each $(g,h) \in G \times G$, $w_{g,h}$ is an invertible element in $D_g D_{gh}$. For all $g, h \in G$:

(P1) $\alpha_e = \mathrm{id}_R$;

(P2) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;

(P3) if $r \in D_{h^{-1}} D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = w_{g,h}\alpha_{gh}(r)w_{g,h}^{-1}$;

(P4) $w_{e,g} = w_{g,e} = 1_g$;

(P5) if $r \in D_{g^{-1}} D_h D_{hl}$, then $\alpha_g(r w_{h,l}) w_{g,hl} = \alpha_g(r) w_{g,h} w_{gh,l}$.

Definition

Given a unital twisted partial action of G on R, we can form the *unital partial crossed product* $R \star_{\alpha}^{w} G = \bigoplus_{g \in G} D_g \delta_g$ where the δ_g's are formal symbols. For $g, h \in G$, $r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

(P6) $(r\delta_g)(r'\delta_h) = r\alpha_g(r' 1_{g^{-1}}) w_{g,h} \delta_{gh}$.

$R \star_{\alpha}^{w} G$ is an associative ring with a natural epsilon-strong G-grading (Nystedt-Öinert-Pinedo).

Theorem (L., Lundström, Öinert, Wagner, 2021)

Suppose that G is torsion-free and that $R \star_{\alpha}^{w} G$ is a unital partial crossed product. Then $R \star_{\alpha}^{w} G$ is prime if and only if R is G-prime.

Theorem (L.,Lundström, Öinert, Wagner, 2021)

The unital partial crossed product $R \star_\alpha^w G$ is not prime if and only if there are:

1. *subgroups $N \lhd H \subseteq G$ with N finite,*
2. *an ideal I of R such that*
   - $\alpha_h(I1_{h^{-1}}) = I1_h$ *for every* $h \in H$,
   - $I \cdot \alpha_g(I1_{g^{-1}}) = \{0\}$ *for every* $g \in G \setminus H$, *and*
3. *nonzero ideals $\tilde{A}, \tilde{B}$ of $R \star_\alpha^w N$ such that $\tilde{A}, \tilde{B} \subseteq I \cdot (R \star_\alpha^w N)$ and $\tilde{A} \cdot 1_h\delta_h \cdot \tilde{B} = \{0\}$ for every $h \in H$.*

Graded prime spectrum of Leavitt path algebras

Definition

A proper graded ideal P of S is called graded prime if for all graded ideals A, B of S, we have $A \subseteq P$ or $B \subseteq P$ whenever $AB \subseteq P$.

1. $\mathrm{Spec}_\gamma(L_k(E))$ is used in the construction of the *algebraic filtered K-theory* by Eilers-Restorff-Ruiz-Sørensen, 2021.
2. $\mathrm{Spec}_\gamma(L_k(E))$ can be described using so-called admissible pairs (I, H) (Larki).

Up next: an alternative description of $\mathrm{Spec}_\gamma(S)$ for S being a general nearly epsilon-strongly G-graded rings.

The graded prime spectrum of nearly epsilon-strongly graded rings

Let S be a G-graded ring and let I be an ideal of S. Define $I_e := I \cap S_e$.

Theorem (L., Lundström, Öinert, Wagner 2021, cf. Nastasescu-van Oystaeyen)

Let S be nearly epsilon-strongly G-graded. The map $I \mapsto I_e$ is a bijection

$$\{\text{graded ideals of } S\} \leftrightarrow \{G - \text{invariant ideals of } S_e\}.$$

Definition

A proper G-invariant ideal Q of S_e is called G-prime if for all G-invariant ideals A, B of S_e , we have $A \subseteq Q$ or $B \subseteq Q$ whenever $AB \subseteq Q$.

The graded prime spectrum of nearly epsilon-strongly graded rings

Theorem (L., Lundström, Öinert, Wagner 2021, cf. Nastasescu-van Oystaeyen)

Let S be nearly epsilon-strongly G-graded. The map $I \mapsto I_e$ is a bijection

$$\{\text{graded prime ideals of } S\} \leftrightarrow \{G - \text{prime ideals of } S_e\}.$$

Question

Is this useful for developing an algebraic filtered K-theory for so-called algebraic Cuntz-Pimsner rings (Carlsen-Ortega), or general nearly epsilon-strongly G-graded rings?

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Cuntz-Pimsner rings

Introduced by Carlsen and Ortega using a categorical approach.

Examples of rings realizable (graded isomorphic) as Cuntz-Pimsner rings:

1. Leavitt path algebras (Carlsen, Ortega 2008)
2. Corner skew Laurent polynomial ring (Carlsen, Ortega 2008)
3. Certain Steinberg algebras (Clark, Fletcher, Hazrat, Li, 2018)

In 2018, Chirvasitu gave new results about the Cuntz-Pimsner C^*-algebra. Inspired by his results I have obtained:

- I. a complete classification of strongly $\mathbb{Z}$-graded algebraic Cuntz-Pimsner rings (Lännström, 2019)
- II. a partial classification of (nearly) epsilon-strongly graded Cuntz-Pimsner rings (Lännström, 2019).

Cuntz-Pimsner rings: Definition

Definition

(Carlsen, Ortega, 2011) Let R be a ring. An *R-system* is a triple (P, Q, ψ) where P and Q are R-bimodules and $\psi \colon P \otimes_R Q \to R$ is an R-bimodule homomorphism from the balanced tensor product $P \otimes_R Q$ to R.

The Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)}$ associated to (P, Q, ψ) is defined as a certain universal object. Note that it is not well-defined for every R-system.

Remark

Let E be a directed graph and K a commutative ring. For a certain choice of (P, Q, ψ) we get that $L_K(E) \cong_{\text{gr}} \mathcal{O}_{(P,Q,\psi)}$.

Classification of strongly graded Cuntz-Pimsner rings

Theorem

(Lännström, 2019) Let $\mathcal{O}_{(P,Q,\psi)}$ be an algebraic Cuntz-Pimsner ring of some system (P, Q, ψ). Then, $\mathcal{O}_{(P,Q,\psi)}$ is unital strongly $\mathbb{Z}$-graded if and only if

$$\mathcal{O}_{(P,Q,\psi)} \cong_{gr} \mathcal{O}_{(P',Q',\psi')}$$

where (P', Q', ψ') is an R'-system satisfying certain conditions.

Remark

The graded isomorphism class of $\mathcal{O}_{(P,Q,\psi)}$ does not uniquely determine (P, Q, ψ). Example of LPA realizable as Cuntz-Pimsner ring in two different ways.

Classification of nearly epsilon-strongly graded Cuntz-Pimsner rings

Theorem

(Lännström, 2019) Let $\mathcal{O}_{(P,Q,\psi)}$ be a Cuntz-Pimsner ring of some system (P,Q,ψ). If $\mathcal{O}_{(P,Q,\psi)}$ is nearly epsilon-strongly $\mathbb{Z}$-graded and $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$, then $\mathcal{O}_{(P,Q,\psi)} \cong_{gr} \mathcal{O}_{(P',Q',\psi')}$ where (P',Q',ψ') is an R'-system and the following assertions are satisfied:

1. (P',Q',ψ') *is an s-unital R'-system;*
2. $(\iota_{P'}^{CP}, \iota_{Q'}^{CP}, \iota_{R'}^{CP}, \mathcal{O}_{(P',Q',\psi')})$ *is a semi-full covariant representation of* (P',Q',ψ');
3. (P',Q',ψ') *satisfies Condition (FS);*
4. $I^{(k)}_{\psi',\iota^{CP}_{\mathcal{O}_0}}$ *is s-unital for $k \geq 0$.*

Conversely, if (P',Q',ψ') is an R'-system satisfying assertions (a)-(d), then $\mathcal{O}_{(P',Q',\psi')}$ is nearly epsilon-strongly $\mathbb{Z}$-graded.

Question

Are there any nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings satisfying $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} \neq \{0\}$? It might be possible to remove this condition from Theorem 95.

Thank you for your attention!