The Noncommutative Geometry of Real Calculi

Axel Tiger Norkvist

Department of Mathematics, Linköping University

June 15, 2023

Linköpings algebraseminarier 2023

Overview

(pseudo-)Riemannian Geometry

(Σ, g) – (pseudo-)Riemannian manifold.

- Σ – Smooth manifold. Locally "indistinguishable" from $\mathbb{R}^n$.
- g – (pseudo-)Riemannian metric. Gives the manifold geometric structure.

At each point $p \in \Sigma$: Tangent space at $p = T_p\Sigma$, basically a copy of $\mathbb{R}^n$ attached to p.

The tangent bundle $T\Sigma = \bigcup_{p\in\Sigma} T_p\Sigma$, disjoint union. $\mathrm{Vect}(\Sigma)$ denotes the module of smooth sections of $T\Sigma$.

Fundamental fact: if $\mathcal{A} = C^\infty(\Sigma)$, then $\mathrm{Der}(\mathcal{A}) \simeq \mathrm{Vect}(\Sigma)$.

The Category (ps-)Rm

Let (ps-)Rm denote "the" category of (pseudo-)Riemannian manifolds.

- Objects: (pseudo-)Riemannian manifolds (Σ, g).
- Morphisms: $\phi : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$, ϕ is a smooth map from Σ_1 to Σ_2 such that the metric is preserved, i.e.,

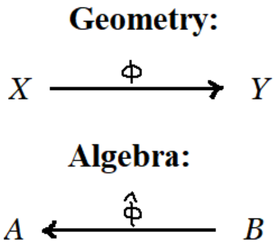
$$\phi^* g_2 = g_1 \iff g_2(\phi_*(X), \phi_*(Y)) = g_1(X, Y), \quad X, Y \in \mathrm{Vect}(\Sigma_1).$$

From Geometric Spaces to Algebras

We note that a morphism $\phi : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$ induces an algebra homomorphism $\hat{\phi} : C^\infty(\Sigma_2) \to C^\infty(\Sigma_1)$ by

$$\hat{\phi}(f)(p) = f(\phi(p)), \quad p \in \Sigma_1.$$

Visually:



Two Central Theorems

Theorem (Gelfand)

Let A be a commutative C^*-algebra. Then there is a locally compact Hausdorff space X such that $A = C_0(X)$.

Theorem (Swan)

Let X be a compact Hausdorff space. Then the category of finitely generated projective modules over the C^*-algebra $C(X)$ of continuous functions on X is equivalent to the category of finite-rank vector bundles on X, where the equivalence is established by sending a vector bundle E to the module of continuous sections of E.

The above theorems give a strong connection between commutative C^*-algebras and geometry, and provide an important conceptual motivation behind NCG.

Riemannian Geometry over Commutative Algebras

Let $\mathcal{A}$ be a commutative *-algebra. Conceptually, we think of it as $C^\infty(X)$ for some unknown, smooth manifold X.

- Q: How to do "(pseudo-)Riemannian geometry" on $\mathcal{A}$?
- (possible) A: Use $\mathrm{Der}(\mathcal{A})$!

Using the natural equivalence between derivations and smooth sections of the tangent bundle, one defines the metric as a symmetric, bilinear map $g : \mathrm{Der}(\mathcal{A}) \times \mathrm{Der}(\mathcal{A}) \to \mathcal{A}$ that is nondegenerate.

The central question

What happens if $\mathcal{A}$ is a noncommutative *-algebra? Can we build a theory of noncommutative geometry in a spirit similar what was done in the commutative case?

An immediate challenge

- $\mathcal{A}$ commutative $\Rightarrow$ $\mathrm{Der}(\mathcal{A})$ has a module structure.
- $\mathcal{A}$ noncommutative $\Rightarrow$ $\mathrm{Der}(\mathcal{A})$ does NOT have a module structure!

Another important difference: if $\mathcal{A}$ is commutative, then every nontrivial derivation ∂ is *outer*, i.e., it cannot be written on the form $\partial(a) = xa - ax$ for some $x \in \mathcal{A}$. If $\mathcal{A}$ is noncommutative, then $\mathrm{Der}(\mathcal{A})$ contains a nontrivial inner derivation for each element that is not central.

How to deal with this?

A Straightforward Approach

From the Serre-Swan theorem: Consider finitely generated projective (right) $\mathcal{A}$-modules as "noncommutative vector bundles".

As for derivations: Choose the derivations of interest, and consider only those.

Metrics

In order to do "geometry" for a noncommutative space, a metric is essential. From the Serre-Swan theorem: define the metric as $h : M \times M \to \mathcal{A}$ for some (right) $\mathcal{A}$-module M.

Classically (when $\mathcal{A}$ is commutative): $h(m_1, m_2) = h(m_2, m_1)$ for all $m_1, m_2 \in M$. In general, this condition is not feasible to include if $\mathcal{A}$ is noncommutative.

Metrics, continued

A metric $h : M \times M \to \mathcal{A}$ is an invertible hermitian form, i.e.,

1. $h(m_1, m_2 + m_3) = h(m_1, m_2) + h(m_1, m_3)$ for $m_1, m_2, m_3 \in M$,
2. $h(m_1, m_2 a) = h(m_1, m_2)a$ for $m_1, m_2 \in M$ and $a \in \mathcal{A}$,
3. $h(m_2, m_1) = h(m_1, m_2)^*$ for $m_1, m_2 \in M$,
4. The map $\hat{h} : M \to M^*$ (where M^* is the dual of M) such that $\hat{h} : m \mapsto h(m, \cdot)$, is invertible.

Affine Connections

We define affine connections. Classically, they can be thought of as "connecting" nearby tangent spaces. We shall view them as "differentiation of sections of a bundle w.r.t tangent vector fields".

Let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ and M projective (right) $\mathcal{A}$-module. A connection $\nabla : \mathfrak{g} \times M \to M$ is such that

1. $\nabla_\partial(m + n) = \nabla_\partial m + \nabla_\partial n$ for $m, n \in M$ and $\partial \in \mathfrak{g}$,
2. $\nabla_{\lambda\partial_1+\partial_2} m = \lambda\nabla_{\partial_1} m + \nabla_{\partial_2} m$ for all $m \in M$, $\lambda \in \mathbb{R}$ and $\partial_1, \partial_2 \in \mathfrak{g}$,
3. $\nabla_\partial(m \cdot a) = (\nabla_\partial m) \cdot a + m \cdot \partial(a)$ for $m \in M$, $\partial \in \mathfrak{g}$ and $a \in \mathcal{A}$.

The Levi-Civita connection (classical case)

The classical case:
A connection $\nabla : \mathrm{Vect}(\Sigma) \times \mathrm{Vect}(\Sigma) \to \mathrm{Vect}(\Sigma)$ is compatible with the metric g if

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \mathrm{Vect}(\Sigma),$$

and torsion-free if

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0, \quad X, Y \in \mathrm{Vect}(\Sigma)$$

Theorem (The fundamental theorem)

Let (Σ, g) be a (pseudo-)Riemannian manifold. Then there exists a unique connection ∇ that is torsion-free and compatible with the metric g.

The connection ∇ is called the Levi-Civita connection.

The noncommutative setting

Can we state a corresponding result to the fundamental theorem for NCG?

Metric compatibility:

$$\partial h(m_1, m_2) = h(\nabla_{\partial^*} m_1, m_2) + h(m_1, \nabla_\partial m_2), \quad \partial \in \mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A}),$$

where $\partial^*(a) = (\partial(a^*))^*$ for $a \in \mathcal{A}$.

What about torsion? An expression

$$\nabla_{m_1} m_2 - \nabla_{m_2} m_1 - [m_1, m_2]$$

makes no sense in the noncommutative setting.

The critical piece

We introduce the notion of anchor maps $\varphi : \mathfrak{g} \to M$, satisfying the following conditions:

1. φ is linear.
2. The module M is generated (as an $\mathcal{A}$-module) by elements of the form $\varphi(\partial)$, $\partial \in \mathfrak{g}$.

Using φ, it is possible to define torsion:

$$T_\varphi(\partial_1, \partial_2) = \nabla_{\partial_1}\varphi(\partial_2) - \nabla_{\partial_2}\varphi(\partial_1) - \varphi([\partial_1, \partial_2]), \quad \partial_1, \partial_2 \in \mathfrak{g}.$$

With this we can talk about Levi-Civita connections in NCG, i.e., connections compatible with the metric h and with vanishing torsion.

Existence and uniqueness of Levi-Civita connections

In general, it is not possible to give a result corresponding to the fundamental theorem of (pseudo-)Riemannian geometry in the noncommutative case.

However, given some further restrictions it is possible to give partial results.

Real calculi, definition

Let A be a unital *-algebra.

Definition (current)

A real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, M, \varphi)$ is a structure such that

1. $\mathfrak{g}$ is a real Lie algebra, and $\pi : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$ faithfully maps elements in $\mathfrak{g}$ to hermitian derivations,
2. M is a (right) $\mathcal{A}$-module,
3. $\varphi : \mathfrak{g} \to M$ is a $\mathbb{R}$-linear map such that M is generated (as an $\mathcal{A}$-module) by elements of the form $\varphi(\partial)$, $\partial \in \mathfrak{g}$.

Extra restrictions

Definition

1. Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, M, \varphi)$ be a real calculus and let $h : M \times M \to \mathcal{A}$ be a metric. The pair $(C_{\mathcal{A}}, h)$ is a real metric calculus if

$$h(\varphi(\partial_1), \varphi(\partial_2)) = h(\varphi(\partial_2), \varphi(\partial_1)), \quad \partial_1, \partial_2 \in \mathfrak{g}.$$

2. If $(C_{\mathcal{A}}, h)$ is a real metric calculus and $\nabla : \mathfrak{g} \times M \to M$ is a connection such that

$$h(\nabla_{\partial_1}\varphi(\partial_2), \varphi(\partial_3))^* = h(\nabla_{\partial_1}\varphi(\partial_2), \varphi(\partial_3)), \quad \partial_1, \partial_2, \partial_3 \in \mathfrak{g},$$

then $(C_{\mathcal{A}}, h, \nabla)$ is a real connection calculus.

3. A real metric calculus $(C_{\mathcal{A}}, h)$ is called pseudo-Riemannian if there exists a Levi-Civita connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is a real connection calculus.

Uniqueness of the Levi-Civita connection

With these restrictions in place, it is possible to prove a uniqueness result for Levi-Civita connections.

Theorem

If $(C_{\mathcal{A}}, h)$ is a pseudo-Riemannian calculus, then there is at most one Levi-Civita connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is a real connection calculus.

Unlike the classical case, the above theorem does not say anything about existence of Levi-Civita connections.

Research questions

There are three main directions that have been considered for real calculi:

1. What classical notions can be given meaning in the context of pseudo-Riemannian calculi?
2. Are there real metric calculi that are not pseudo-Riemannian?
3. When are two real (metric) calculi indistinguishable as algebraic structures?

The noncommutative torus

The noncommutative torus T^2_θ is the *-algebra with unitary generators U, V satisfying the relation $VU = qUV$, where $q = e^{2\pi i\theta}$. Choose derivations δ_1, δ_2 given by:

$$\delta_1(U) = iU \qquad \delta_2(U) = 0$$
$$\delta_1(V) = 0 \qquad \delta_2(V) = iV.$$

We have that $[\delta_1, \delta_2] = 0$.

In analogy with the classical torus T^2 being parallellizable, we let M' be a free module of rank 2, with basis e'_1, e'_2. With $\varphi'(\delta_i) = e'_i$ $(i = 1, 2)$, the real calculus $C_{T^2_\theta} = (T^2_\theta, \mathfrak{g}'_{\pi'}, M', \varphi')$ over T^2_θ represents the noncommutative torus.

The noncommutative 3-sphere

The noncommutative 3-sphere S^3_θ is the unital *-algebra with generators Z, Z^*, W, W^* subject to the relations

$$\begin{aligned} WZ &= qZW & W^*Z &= \bar{q}ZW^* & WZ^* &= \bar{q}Z^*W \\ W^*Z^* &= qZ^*W^* & Z^*Z &= ZZ^* & W^*W &= WW^* \\ WW^* &= \mathbb{1} - ZZ^*, \end{aligned}$$

Choose derivations $\partial_1, \partial_2, \partial_3$ given by:

$$\begin{aligned} \partial_1(Z) &= iZ, & \partial_2(Z) &= 0, & \partial_3(Z) &= ZWW^* \\ \partial_1(W) &= 0 & \partial_2(W) &= iW & \partial_3(W) &= -WZZ^*. \end{aligned}$$

We have that $[\partial_i, \partial_j] = 0$ for all $i, j = 1, 2, 3$. In analogy with the 3-sphere S^3 being parallellizable, let M be a free module of rank 3 with basis e_1, e_2, e_3. Let $\varphi(\partial_i) = e_i$ $(i = 1, 2, 3)$. Then the real calculus $C_{S^3_\theta} = (S^3_\theta, \mathfrak{g}_\pi, M, \varphi)$ over S^3_θ represents the noncommutative 3-sphere.

Embeddings

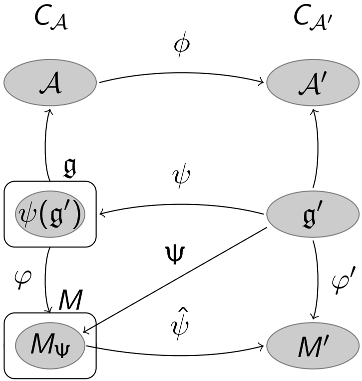
Let $\phi : (\Sigma_1, g_1) \hookrightarrow (\Sigma_2, g_2)$ be an isometric embedding of Σ_1 into Σ_2. Then ϕ can be viewed as a morphism in (ps-)Rm with extra structure:

1. ϕ is an injective immersion, i.e., its pushforward $\phi_* : \mathrm{Vect}(\Sigma_1) \to \mathrm{Vect}(\Sigma_2)$ is everywhere injective.
2. Σ_1 is diffeomorphic to $\phi(\Sigma_1) \subset \Sigma_2$.

Let (C_1, h_1) and (C_2, h_2) be real metric calculi corresponding to (Σ_1, g_1) and (Σ_2, g_2), respectively. To describe the embedding of (C_1, h_1) into (C_2, h_2) we shall define a morphism $(C_2, h_2) \to (C_1, h_1)$ of real metric calculi.

Real Calculus Homomorphism, illustration

A schematic picture of $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$:



Compatibility conditions

1. $\delta(\phi(a)) = \phi(\psi(\delta)(a))$
2. $\hat{\psi}(m \cdot a) = \hat{\psi}(m) \cdot \phi(a)$
3. $\hat{\psi}(\varphi(\psi(\delta))) = \varphi'(\delta)$

Extra compatibility condition for real metric calculus morphisms $(\phi, \psi, \hat{\psi}) : (C_{\mathcal{A}}, h) \to (C_{\mathcal{A}'}, h')$

$$h'(\varphi'(\partial'_1), \varphi'(\partial'_2)) = \phi(h(\Psi(\partial'_1), \Psi(\partial'_2))).$$

Embeddings of Real Calculi

Definition

A real calculus homomorphism $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is called an embedding of $C_{\mathcal{A}'}$ into $C_{\mathcal{A}}$ if ϕ is surjective and there is a submodule $\tilde{M}$ of M such that $M = M_{\Psi} \oplus \tilde{M}$. Moreover, if $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ are real metric calculi such that $h'(\hat{\psi}(m_1), \hat{\psi}(m_2)) = \phi(h(m_1, m_2))$ for all $m_1, m_2 \in M_{\Psi}$ and $M = M_{\Psi} \oplus M_{\Psi}^{\perp}$ (w.r.t h), then we say that $(C_{\mathcal{A}'}, h')$ is isometrically embedded into $(C_{\mathcal{A}}, h)$ by $(\phi, \psi, \hat{\psi})$, and h' is called the induced metric.

Orthogonal decomposition of ∇

Let $(C_\mathcal{A}, h)$ and $(C_{\mathcal{A}'}, h')$ be pseudo-Riemannian calculi (with Levi-Civita connections ∇ and ∇', resp.) such that $(\phi, \psi, \hat{\psi}) : C_\mathcal{A} \to C_{\mathcal{A}'}$ is an isometric embedding of $(C_{\mathcal{A}'}, h')$ into $(C_\mathcal{A}, h)$.

Let $m \in M_\Psi$ and let $\xi \in M_\Psi^\perp$. One may split ∇ into tangential and normal parts in the following way:

$$\nabla_{\psi(\delta)} m = L(\delta, m) + \alpha(\delta, m) \quad \text{(Gauss' formula)}$$
$$\nabla_{\psi(\delta)} \xi = -A_\xi(\delta) + D_\delta \xi \quad \text{(Weingarten's formula);}$$

$\alpha : \mathfrak{g}' \times M_\Psi \to M_\Psi^\perp$ is called the second fundamental form, and $A : \mathfrak{g}' \times M_\Psi^\perp \to M_\Psi$ is called the Weingarten map.

Free real calculi

Let $C_{\mathcal{A}} = (\mathcal{A}, g_\pi, M, \varphi)$ be a real calculus where $\mathfrak{g}$ has basis $\partial_1, ..., \partial_n$, which is such that

1. $M \simeq \mathcal{A}^n$ is free,
2. the set $\varphi(\partial_1), ..., \varphi(\partial_n)$ forms a basis of M.

Then $C_{\mathcal{A}}$ is called a free real calculus.

- If $(C_{\mathcal{A}}, h)$ is a free real metric calculus, then it is also pseudo-Riemannian. Conceptually, a free real metric calculus can be thought of as a parallellizable manifold.
- $C_{T^2_\theta}$ and $C_{S^3_\theta}$ given earlier are both free.
- Mean curvature can be defined for embeddings of free real metric calculi in a straightforward manner.

Mean curvature and minimality of an embedding

Definition

Let $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ be pseudo-Riemannian real calculi such that $(C_{\mathcal{A}'}, h')$ is free, and let $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ be an isometric embedding of $(C_{\mathcal{A}'}, h')$ into $(C_{\mathcal{A}}, h)$.

For a basis $\delta_1, ..., \delta_k$ of $\mathfrak{g}'$, the mean curvature $H_{\mathcal{A}'} : M \to \mathcal{A}'$ is:

$$H_{\mathcal{A}'}(m) = \phi(h(m, \alpha(\delta_i, \Psi(\delta_j))))(h')^{ij}, \quad m \in M.$$

- The value of $H_{\mathcal{A}'}(m)$ is independent of the choice of basis $\delta_1, ..., \delta_k$ for all $m \in M$,
- $H_{\mathcal{A}'}(m) = 0$ for all $m \in M_\Psi$,
- We say that an embedding is minimal if the mean curvature is zero, i.e. $H_{\mathcal{A}'}(m) = 0$ for all $m \in M$.

The noncommutative torus again

The noncommutative torus T^2_θ is the *-algebra with unitary generators U, V satisfying the relation $VU = qUV$, where $q = e^{2\pi i\theta}$.
Choose derivations δ_1, δ_2 given by:

$$\delta_1(U) = iU \qquad \delta_2(U) = 0$$
$$\delta_1(V) = 0 \qquad \delta_2(V) = iV.$$

We have that $[\delta_1, \delta_2] = 0$.

In analogy with the classical torus T^2 being parallellizable, we let M' be a free module of rank 2, with basis e'_1, e'_2. With $\varphi'(\delta_i) = e'_i$ $(i = 1, 2)$, the real calculus $C_{T^2_\theta} = (T^2_\theta, \mathfrak{g}'_{\pi'}, M', \varphi')$ over T^2_θ represents the noncommutative torus.

The noncommutative 3-sphere again

The noncommutative 3-sphere S^3_θ is the unital *-algebra with generators Z, Z^*, W, W^* subject to the relations

$$
\begin{aligned}
WZ &= qZW & W^*Z &= \bar{q}ZW^* & WZ^* &= \bar{q}Z^*W \\
W^*Z^* &= qZ^*W^* & Z^*Z &= ZZ^* & W^*W &= WW^* \\
WW^* &= \mathbb{1} - ZZ^*,
\end{aligned}
$$

Choose derivations $\partial_1, \partial_2, \partial_3$ given by:

$$
\begin{aligned}
\partial_1(Z) &= iZ, & \partial_2(Z) &= 0, & \partial_3(Z) &= ZWW^* \\
\partial_1(W) &= 0 & \partial_2(W) &= iW & \partial_3(W) &= -WZZ^*.
\end{aligned}
$$

We have that $[\partial_i, \partial_j] = 0$ for all $i, j = 1, 2, 3$. In analogy with the 3-sphere S^3 being parallellizable, let M be a free module of rank 3 with basis e_1, e_2, e_3. Let $\varphi(\partial_i) = e_i$ $(i = 1, 2, 3)$. Then the real calculus $C_{S^3_\theta} = (S^3_\theta, \mathfrak{g}_\pi, M, \varphi)$ over S^3_θ represents the noncommutative 3-sphere.

A minimal embedding of T^2_θ into S^3_θ

An embedding of T^2_θ into S^3_θ is achieved by the *-homomorphism $\phi : S^3_\theta \to T^2_\theta$ given by $\phi(Z) = \lambda U$ and $\phi(W) = \mu W$, where λ and μ are nonzero complex constants such that $|\lambda|^2 + |\mu|^2 = 1$.

With this choice of ϕ, we have ψ given by $\psi(\delta_i) = \partial_i$ and $\hat{\psi}$ is then given by $\hat{\psi}(e_i) = e'_i$ for $i = 1, 2$, and we have that $(\phi, \psi, \hat{\psi}) : C_{S^3_\theta} \to C_{T^2_\theta}$ is an embedding of the noncommutative torus into the noncommutative 3-sphere.

Proposition

Let h be the standard metric on the noncommutative 3-sphere, i.e., $h(e_1, e_1) = ZZ^*, h(e_2, e_2) = WW^*, h(e_3, e_3) = ZZ^*WW^*$, and $h(e_i, e_j) = 0$ if $i \neq j$, and let h' be the induced metric on the noncommutative torus. Then the above embedding is minimal if $|\lambda| = |\mu| = 1/\sqrt{2}$.

Free and Projective Real Calculi

A real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, M, \varphi)$ is called projective if M is projective.

- By the Serre-Swan theorem, projective real calculi are especially interesting to study.
- Every free real calculus is projective.
- If M is free, this does not necessarily imply that $C_{\mathcal{A}}$ is free.

Given a free real calculus $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, \mathcal{A}^n, \tilde{\varphi})$ and a projection $P : \mathcal{A}^n \to \mathcal{A}^n$, then the real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, P(\mathcal{A}^n), P \circ \tilde{\varphi})$ can be seen as a "projection" of $\tilde{C}_{\mathcal{A}}$.

Free and Projective Real Calculi, continued

Proposition

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, M, \varphi)$ be a projective real calculus. Then there exists a free real calculus $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_\pi, \mathcal{A}^n, \tilde{\varphi})$ and a projection $P : \mathcal{A}^n \to \mathcal{A}^n$ such that $C_{\mathcal{A}}$ is isomorphic to $(\mathcal{A}, \mathfrak{g}_\pi, P(\mathcal{A}^n), P \circ \tilde{\varphi})$.

The above proposition can be used to find objects on projective real calculi by defining them on a free real calculus and then project them down.

Free vs. Projective Real Calculi

In a sense, free real calculi are very easy to work with.

- If $(C_{\mathcal{A}}, h)$ is a free real metric calculus it is automatically pseudo-Riemannian as well.
- If the real calculi $(\mathcal{A}, \mathfrak{g}_\pi, M, \varphi)$ and $(\mathcal{A}, \mathfrak{g}_\pi, M, \varphi')$ are free, then they are isomorphic.

For general projective real calculi the situation is far more interesting (and difficult).

Real Calculi over Matrix Algebras

Let $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$ for some N. We consider the module $M = \mathbb{C}^N$, which is projective and simple. All derivations on $\mathcal{A}$ are inner, and hermitian derivations correspond to unique elements of $\mathfrak{su}(N)$ of anti-hermitian traceless matrices. Therefore, let

- $\mathfrak{g} \subseteq \mathfrak{su}(N)$, with basis $D_1, ..., D_n$.
- $\pi : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$, given by $\pi(D_i) = \partial_i = [D_i, \cdot]$.

Note that since $\mathbb{C}^N$ is simple, any nonzero $\mathbb{R}$-linear map $\varphi : \mathfrak{g} \to \mathbb{C}^N$ is an anchor map.

The case $\mathfrak{g} = \mathbb{R}\langle D \rangle \subset \mathfrak{su}(N)$

Let $\mathfrak{g} = \mathbb{R}\langle D \rangle \subset \mathfrak{su}(N)$, the 1-dimensional Lie algebra generated by D, and let $C_{\mathcal{A}} = (\text{Mat}_N(\mathbb{C}), \mathfrak{g}_\pi, \mathbb{C}^N, \varphi)$ be a real calculus. If D is fixed, then it is possible to calculate the exact number of nonisomorphic real calculi of the form $(\text{Mat}_N(\mathbb{C}), \mathfrak{g}_\pi, \mathbb{C}^N, \varphi)$, where φ is arbitrary.

Proposition

Let k be the number of distinct eigenvalues of D and let $|C_D|$ denote the number of pairwise nonisomorphic real calculi of the form $(\text{Mat}_N(\mathbb{C}), \mathfrak{g}_\pi, \mathbb{C}^N, \varphi)$. Then

1. if D is not anti-selfsimilar, then $|C_D| = 2^k - 1$,
2. if D is anti-selfsimilar and k is odd, then $|C_D| = 2^{(k-1)/2}(1 + 2^{(k-1)/2}) - 1$,
3. if D is anti-selfsimilar and k is even, then $|C_D| = 2^{k/2-1}(1 + 2^{k/2}) - 1$.

Metric Anchor Maps for the Matrix Example

With $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$ and $M = \mathbb{C}^N$, all metrics $h : M \times M \to \mathcal{A}$ are of the form

$$h(u, v) = x \cdot u^\dagger v, \quad u, v \in \mathbb{C}^N, \quad x \in \mathbb{R} \setminus \{0\},$$

where $\dagger$ denotes the hermitian transpose. Moreover, the matrix $u^\dagger v$ is self-adjoint if and only if $u = \mu \cdot v$ for some $\mu \in \mathbb{R}$.

Let $D_1, \ldots D_n$ be a basis of $\mathfrak{g}$ and let $\varphi : \mathfrak{g} \to \mathbb{C}^N$ be a metric anchor map. Since this implies that

$$h(\varphi(D_i), \varphi(D_j)) = h(\varphi(D_i), \varphi(D_j))^\dagger = x \cdot \varphi(D_i)^\dagger \varphi(D_j),$$

which implies that there is a unit vector $\hat{v}_0 \in \mathbb{C}^N$ and $\mu_1, \ldots, \mu_n \in \mathbb{R}$ such that

$$\varphi(D_i) = \mu_i \hat{v}_0.$$

Connections for the Matrix Example

To characterize a generic connection ∇ on $\mathfrak{g} \times M$, we note that $\nabla_i = \nabla_{\partial_i}$ is a linear map for each i. Hence,

$$\nabla_i v = v \cdot X_i, \quad v \in \mathbb{C}^N$$

for a unique matrix X_i. The Leibniz condition for ∇ implies that

$$\begin{aligned}(v \cdot A) \cdot X_i = \nabla_i(v \cdot A) &= (\nabla_i v) \cdot A + v \cdot \partial_i(A) \\ &= v \cdot (X_i A) + v \cdot [D_i, A], \quad v \in \mathbb{C}^N, A \in \mathrm{Mat}_N(\mathbb{C}).\end{aligned}$$

This implies that $X_i = t_i \mathbb{1}_N - D_i$ or, equivalently, that

$$\nabla_i v = t_i v - v \cdot D_i,$$

where $t_i \in \mathbb{C}$ are arbitrary complex constants. The connection is compatible with the metric h iff $t_j = i\lambda_j$ for $\lambda_j \in \mathbb{R}$, i.e., if X_j is antihermitian.

The case $N = 2$ and $\mathfrak{g} = \mathfrak{su}(2)$

In the matrix example, let $N = 2$ and let $\mathfrak{g} = \mathbb{R}\langle D_1, D_2, D_3\rangle = \mathfrak{su}(2)$, given by

$$D_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The maps φ we consider are of the form $\varphi(\partial_i) = \mu_i \hat{v}_0$, where $\hat{v}_0 \in \mathbb{C}^2$ is nonzero and $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are not all zero, i.e., φ is a metric anchor map.

With $X_j = t_j \mathbb{1} - D_j$ we have that $\nabla_j v = v \cdot X_i$ defines a metric connection iff each $t_j = i\lambda_j$, $\lambda_j \in \mathbb{R}$, implying that X_i is skew-hermitian.

The matrix example, part 2 (continued)

Letting $f^i_{jk} \in \mathbb{R}$ denote the structure constants of $\mathfrak{g}$ (i.e., $[\partial_i, \partial_j] = f^k_{ij}\partial_k$), we get the torsion T_φ to be

$$T_\varphi(D_i, D_j) = \nabla_i\varphi_j - \nabla_j\varphi_i - \varphi([\partial_i, \partial_j]) = \hat{v}_0(\mu_j X_i - \mu_i X_j - f^k_{ij}\mu_k \mathbb{1}).$$

This expression vanishes for all i, j iff $\hat{v}_0$ is an eigenvector of $T_{ij} = \mu_j X_i - \mu_i X_j - f^k_{ij}\mu_k\mathbb{1}$ with eigenvalue $\lambda_{ij} = 0$. Noting that $\Re(\lambda_{ij}) = -f^k_{ij}\mu_k$, it follows that $-f^k_{ij}\mu_k = 0$. However, solving these equations for all i, j, we get

$$\Re(\lambda_{12}) = 2\mu_3 = 0, \quad \Re(\lambda_{13}) = -2\mu_2 = 0, \quad \Re(\lambda_{23}) = 2\mu_1 = 0.$$

This implies that $\varphi \equiv 0$, which is not an anchor map. We have thus proven that if $((\mathrm{Mat}_2(\mathbb{C}), \mathfrak{su}(2)_\pi, \mathbb{C}^2, \varphi), h)$ is a real metric calculus, then it is not pseudo-Riemannian.

The general statement for $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$ and $M = \mathbb{C}^N$

The general statement for $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$ and $M = \mathbb{C}^N$ is as follows.

Theorem

There exists a metric anchor map $\varphi : \mathfrak{g} \to \mathbb{C}^N$ such that the resulting real metric calculus $((\mathrm{Mat}_N(\mathbb{C}), \mathfrak{g}_\pi, \mathbb{C}^N, \varphi), h)$ is pseudo-Riemannian if and only if $\mathfrak{g} \subseteq \mathfrak{su}(N)$ is not semisimple and there exists a common eigenvector to all matrices in $\mathfrak{g}$.

The End