Introduction

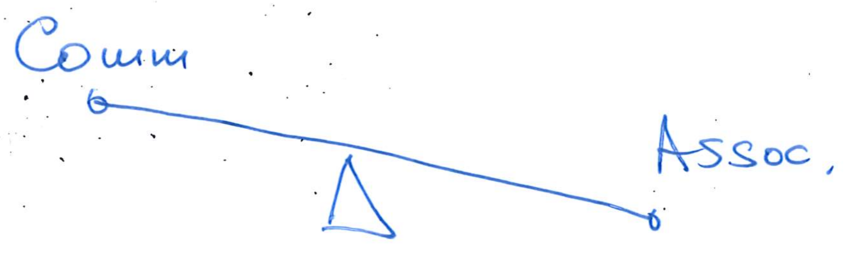
Appearance of the Complexes $\mathbb{C}$ was related to very practical things: solutions of quadratic equations like $x^2+2=0$.
It was absolutely unexpected from this point that $\mathbb{C}$ are important for many classical developments like analytic functions, conformal mappings etc. Still one can claim about "unreasonable effectiviness of $\mathbb{C}$" (E. Wigner)

- Eugene Wigner (1902-1995) Hungarian, German, America

- Von Neumann (Janos Lajos Neumann) 1903-1957

- Pascual Jordan (1902-1980) German
  ("Jorda") Spanish origin., after Napoleon

Comm $\longleftrightarrow$ Ass: <u>swing</u> (gunga)



PBW (Birkhoff-Witt) $\to$ <u>enveloping algebra</u>

Basic facts about Jordan algebras

- $[L(x), L(x^2)] = 0$, $xy = yx$. (1)
- $L(x^3) - 3L(x^2)L(x) + 2L(x)^3 = 0$ (2)
- Any Jordan algebra is power associative:

□ (2) $\Rightarrow$ $x^3x - 3x^2x^2 + 2x^3x = 0 \Rightarrow x^3x = x^2x^2$ (3)

Also $x^3x^2 - 3x^2x^3 + 2x^5 = 0$ $\underline{x^5 = x^2x^3}$ (4)

$\Rightarrow (x^2x^2)x \overset{(3)}{=} x(x^4) = x^2x^3$ etc. □

Special algebras.

If A is associative then A^+ is Jordan.

A^- is Lie.

□ $x \circ y = xy + yx$, $x^{\circ 2} = x^2$

$$x^2 \circ (x \circ y) = \underbrace{x^2(xy) + x^2(yx) + (xy)x^2}_{\text{assoc.}} + (yx)x^2$$

$$= x(x^2y) + (x^2y)x + x(yx^2) + (yx^2)x =$$

$$= x(x^2 \circ y) + (x^2 \circ y)x = (x^2 \circ y) \circ x.$$ □

If $(A, *)$ associative with involution then $Herm(A, *)$ with Jordan product are a Jordan algebra $=$ archetypes of all Jordan algebras (formally real).

$x^n - p_1(x)x^{n-1} + \dots + p_n(x)e = 0$ minimal polynomi.

$\therefore p_1(x) = \operatorname{tr} x$! $\operatorname{tr}(xy)z = \operatorname{tr} x(yz)$

Spin factors ("Clifford type")

Let V be a vector space with bilinear form $\langle , \rangle : V \times V \to \mathbb{F}$, consider $A = \mathbb{F} \times V =: \mathbb{F} \oplus V$

$$(a,u)\cdot(b,v) = (ab + \langle u,v\rangle, av + bu)$$

Ex. 1 $\mathbb{R}\oplus\mathbb{R}$ with $\langle u,v\rangle = uv$ is $\cong \mathbb{C}$. But $\mathbb{C}\oplus\mathbb{C} \not\cong \mathbb{H} \leftarrow$ noncommutative

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Ex. 2 $\mathbb{R}\oplus\mathbb{R}^2$: $(a,u_1,u_2)^2 = (a^2+u_1^2+u_2^2, 2au_1, 2au_2)$

$$\xi = (a,u_1,u_2) \xrightarrow{\varphi} \begin{pmatrix} a+u_1 & u_2 \\ u_2 & a-u_1 \end{pmatrix}, \quad \det(\varphi(\xi)) = a^2 - u_1^2 - u_2^2$$

$$\varphi(\xi) = aI + u_1 E + u_2 F, \quad \underbrace{I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{analogues to } sl_2.}$$

Note

$\bullet$	I	E	F
I	I	E	F
E	E	I	0
F	F	0	I

$$\varphi(\xi)\bullet\varphi(\xi) = (a^2+u_1^2+u_2^2)I + 2u_1 a E + 2u_2 a F$$

Hence

- $\boxed{\underset{\text{spin}}{\mathbb{R}\oplus\mathbb{R}^2} \cong \mathrm{Herm}(\mathbb{R},2)}$ Similar
- $\mathbb{R}\oplus\mathbb{R}^3 \cong \mathrm{Herm}(\mathbb{C},2)$
- $\mathbb{R}\oplus\mathbb{R}^5 \cong \mathrm{Herm}(\mathbb{H},2)$
- $\mathbb{R}\oplus\mathbb{R}^9 \cong \mathrm{Herm}(\mathbb{O},2)$

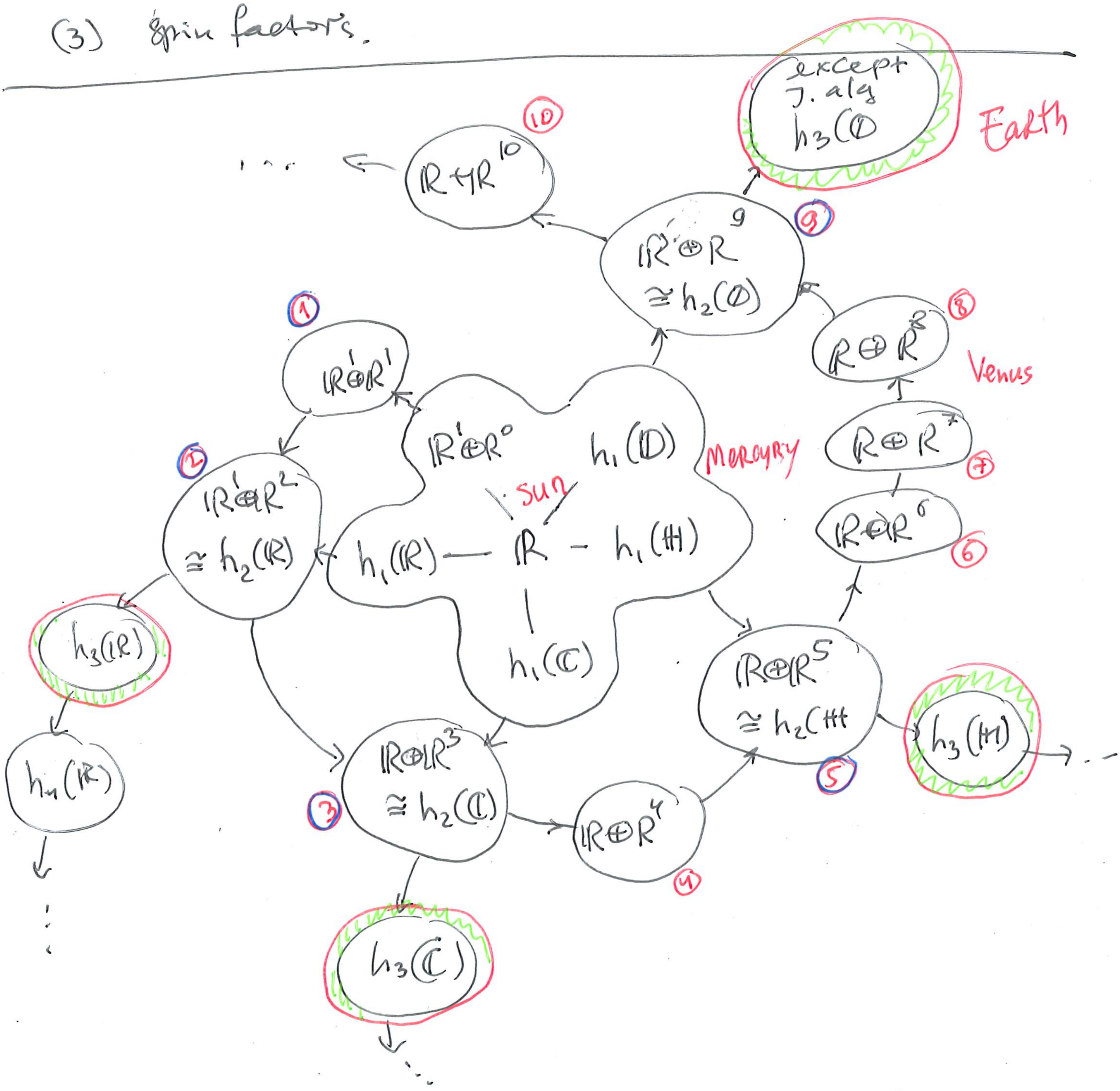
Jordan-Von Neumann-Wigner

Every finite-dim. formally real Jordan algebra is a direct sum of a finite number of simple. There ar 5 types of simple algebras:

(1) $\mathrm{Herm}(\mathbb{F}_d, n)$ $d=1,2,4$ and $n \geq 2$

(2) $\mathrm{Herm}(\mathbb{F}_8, 3)$ Albert

(3) spin factors.



Cubic forms

- $N: V \to \Phi$ is a homogeneous polynomial of $\deg N = 3$.
- $N(x,y,z) = N(x+y+z) - N(x+y) - N(y+z) - N(z+x) + N(x) + N(y) + N(z)$
- $N(x,x,x) = (27 - 3\cdot 8 + 3) N(x) = 6N(x)$
- $N(x;y) = \frac{1}{2} N(x,x,y) = \partial_y N(x)$

A Basepoint for N : $c \in V$ and $N(c) = 1$

Given c, define

- $N(c,x) = T(x)$ (trace form) $= \frac{1}{2} N(c,c,x)$
- $N(x,c) = S(x)$ (spure form) $= \frac{1}{2} N(x,x,c)$

Linearization

$$S(x,y) = S(x+y) - S(x) - S(y) = N(x,y,c)$$

" think of $N(x) = x^3$, $N(x,y) = 3x^2y$, $N(x,y,z) = 6xyz$.

Def Sharp (adjoint) map $\#: V \to V$ defined by

$$T(x^\#; y) = N(x; y)$$

where $T(x,y) = T(x)T(y) - S(x,y) =$

$$= N(c,x)N(c,y) - N(c,x,y)$$

$\boxed{\text{" torsion "}}$ $\nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$?

Def (N,c) over Φ is Jordan cubic if

1) N iss nondegenerate at c (T is nondegenerate)

2) $(x^\#)^\# = N(x)x$ (adjoit identity holds)

Springer construction.

From every Jordan cubic form with basepoint e we obtain a Jordan algebra $J(N,e)$:

1) with unit $= e$
2) product: $x \cdot y := \frac{1}{2}\left(x \# y + T(x)y + T(y)x - S(x,y)e\right)$
3) of degree 3:
$$x^{\cdot 3} - T(x)\,x^{\cdot 2} + S(x)\,x - N(x)\,e = 0$$
4') $x^{\#} = x^{\cdot 2} - T(x)x + S(x)e$

Example $\mathrm{Mat}(\mathbb{F},3)^{+}$ (symmetric matriser över $\mathbb{F}$)

- $N(x) = \det x$, $e = I$.
- $\det x = \frac{1}{3}\operatorname{tr} x^3 - \frac{1}{2}\operatorname{tr} x \operatorname{tr} x^2 + \frac{1}{6}(\operatorname{tr} x)^3 =: N(x)$

Linearization: • $N(x;y) = \operatorname{tr} x^2 y - \frac{1}{2}\operatorname{tr} y \operatorname{tr} x^2 - \operatorname{tr} x \operatorname{tr} xy + \frac{1}{2}(\operatorname{tr} x)^2 \operatorname{tr} y$

- $N(x,y,z) = \operatorname{tr}(xzy + xyz) - \sum \operatorname{tr} x \operatorname{tr} yz + \operatorname{tr} x \operatorname{tr} y \operatorname{tr} z$
- $T(y) = N(e;y) = \operatorname{tr} y - \frac{3}{2}\operatorname{tr} y - 3\operatorname{tr} y + \frac{9}{2}\operatorname{tr} y = \operatorname{tr} y$
- $N(x;e) = \operatorname{tr} x^2 - \frac{3}{2}\operatorname{tr} x^2 - (\operatorname{tr} x)^2 + \frac{3}{2}(\operatorname{tr} x)^2 = \frac{(\operatorname{tr} x)^2 - \operatorname{tr} x^2}{2}$
- $N(x,y,e) = \operatorname{tr} x \operatorname{tr} y - \operatorname{tr}(xy)$
- $T(x,y) = T(x)T(y) - N(x,y,e) = \operatorname{tr}(xy)$ – nondegenerate.
- $x^{\#}$: $T(x^{\#},y) = N(x;y) = \operatorname{tr}\left(\left(x^2 - \frac{1}{2}\operatorname{tr} x^2\, I - \operatorname{tr} x \cdot x + \frac{1}{2}(\operatorname{tr} x)^2 I\right) y\right)$

$$\Rightarrow \quad x^{\#} = x^2 - x \operatorname{tr} x + \frac{(\operatorname{tr} x)^2 - \operatorname{tr} x^2}{2} I \quad \leftarrow \text{adjoint} \qquad (*)$$

Indeed $x^{\#}x = x^3 - x^2 \operatorname{tr} x + \frac{(\operatorname{tr} x)^2 - \operatorname{tr} x^2}{2} x = N(x)\, I$

In other words $\underline{x^{\#} = N(x)\, x^{-1}}$ for almost all x

$\Rightarrow x^{\#\#} = N(N(x)x^{-1})(N(x)x^{-1})^{-1} = \frac{N^3(x)}{N(x)} \cdot \frac{1}{N(x)} \cdot x = N(x)\,x.$

$\Rightarrow$ Joran cubic form. Then

$x \cdot y = \frac{1}{2}(x\#y + y\,\mathrm{tr}x + x\,\mathrm{tr}y - (\mathrm{tr}x\,\mathrm{tr}y - \mathrm{tr}(xy))\,c) \ominus$

$x\#y = (x+y)^{\#} - x^{\#} - y^{\#} =$ by $(*)$

$= (xy + yx) - (x\,\mathrm{tr}y + y\,\mathrm{tr}x) + (\mathrm{tr}x\,\mathrm{tr}y - \mathrm{tr}xy)\,c)$

$\ominus\ \frac{1}{2}(xy+yx) \leftarrow$ Jordan multiplication !

Remark There are one isomorphy class of vector spaces in $\dim = n$, but $\exists$ a huge member of different problem reduced to it. The same for Jord.

Eiconal Algebras (from isoparametric hypersfs)

Let $\mathbb{R}^n$ be a vector space, $x \in \mathbb{R}^n$, $u(x)$ be a homogeneous form of degree $g \geq 2$, $u(x) =:$ isoparametric if

(1) $|\nabla u(x)|^2 = c_1 |x|^{2g-2}$

(2) $\Delta u(x) = c_2 |x|^{g-2}$ (0 if m odd)

A deep result states that $g \in \{1, 2, 3, 4, 6\}$.

Geometric context: leves sets $\{u(x) = t\} \cap \{|x| = 1\}$ are compact hypersurfaces of $\{|x| = 1\}$ with g dist. constant principal curvatures. Each curvature has multiplicity m_1 or m_2, where:

$$\begin{cases} |\nabla u|^2 = g^2 |x|^{2g-2} & \text{(i)} \\ \Delta u = \frac{m_1 - m_2}{2} g^2 |x|^{g-2} & \text{(ii)} \end{cases}$$

Case $g = 3$, without (ii) is equivalent to eiconal equation of special type.

$$|\nabla u(x)|^2 = 9 |x|^4, \quad x \in V = \mathbb{R}^n \qquad \text{(B1)}$$

In general, let $b(x, y)$ be a bilinear form on V.

~~Then~~

$$b(\nabla u(x), \nabla u(x)) = 9\, b(x, x)^2, \qquad \text{(B2)}$$

b being a non-degenerate form on a vector sp. V.

$\Rightarrow$ $(V, b, u) =$ eiconal triple. over field $\mathbb{F}$

$Q(x) = b(x, x)$

Gradient: here is defined by "duality":

$\nabla u(x)$ is the unique by $b(\nabla u(x); y) = u(x; y)$, $\forall y \in V$

Define an auxiliary multiplicative structure $*$ on V by
$u(x;y;z) = b(x*y; z)$ (uniquely determined!)

Eqivalently: $x*y = \nabla u(x+y) - \nabla u(x) - \nabla u(z)$ (explicitly)

In this notation $x*x = 4\nabla u(x) - 2\nabla u(x) = 2\nabla u(x)$

$\Rightarrow$ $\boxed{\nabla u(x) = \frac{1}{2} x*x}$

Then the eiconal eq. (B2) becomes $\boxed{b(x*x, x*x) = 36\, b^2(x,x)}$ (B3)
an algebra identity.

Hence: $u(x;y) = b(\nabla u(x); y) = \frac{1}{2} b(x*x, y)$

$u(x;x) = \frac{1}{2} b(x*x, x) \equiv \frac{\partial u(x)}{\partial x} = 3u(x)$

$\Rightarrow$ $u(x) = \frac{1}{6} b(x*x, x)$ and (B4)

$u(x;y;z) = b(x; y*z) = b(y, x*z) = \ldots$
(metrized)

Now, starting with a given cubic form $u(x)$ on a vector space V over $\mathbb{F}$, we shall define a new cubic form N on $\hat{V} = \mathbb{F} \times V$ such that the new form is Jordan, and the original form $u(x)$ is obtained from N by reduction.

So let $\hat{V} = \mathbb{F} \times V$, $c = (1,0)$, $\hat{x} = (x_0, x)$

$$\boxed{N(\hat{x}) := x_0^3 - \frac{3}{2} x_0 b(x,x) + \frac{1}{\sqrt{2}} u(x)}$$

(B4) $\frac{1}{2} b(x*x, y)$

Then

1) $N(\hat{x};\hat{y}) = 3x_0^2 y_0 - \frac{3}{2} y_0 b(x,x) - 3x_0 b(x,y) + \frac{1}{\sqrt{2}} u(x;y)$

$$= 3\left(x_0^2 - \frac{1}{2} b(x,x)\right) y_0 + b\left(\frac{x*x}{2\sqrt{2}} - 3x_0 x; y\right)$$

$S(\hat{x}) = N(\hat{x}, c)$

2) $N(\hat{x},\hat{y},\hat{z}) = 6x_0 y_0 z_0 - 3\left(b(x,z) y_0 + b(y,z) x_0 + b(x,y) z_0\right)$

$$+ \frac{1}{\sqrt{2}} b(x*y, z)$$

Trace
- $T(\hat{x}) = N(c;\hat{x}) = 3x_0$
- $N(\hat{x},\hat{y},c) = 6x_0 y_0 - 3b(x,y)$
- $T(\hat{x},\hat{y}) = T(\hat{x})T(\hat{y}) - N(\hat{x},\hat{y},c) =$

  $= 3(x_0 y_0 + b(x,y)) \leftarrow$ nondegenerate !
- $N(\hat{x},\hat{y}) = T(\hat{x}^{\#},\hat{y})$ (see def, p.5). Then

$$\begin{cases} (\hat{x}^{\#})_0 = x_0^2 - \frac{1}{2} b(x,x) =: z_0 \\ (\hat{x}^{\#})_V = \frac{x*x}{6\sqrt{2}} - x_0 x =: z \end{cases}$$

$$\hat{z} = \hat{x}^{\#}$$

We want to prove: $(\hat{x}^{\#})^{\#} = N(x)\hat{x}$!

$(w_0, w) := \hat{x}^{\#\#} = \left(z_0^2 - \frac{1}{2} b(z,z);\ \frac{z*z}{6\sqrt{2}} - z_0 z\right)$. Then

(a) $w_0 = \left(x_0^2 - \frac{1}{2} b(x,x)\right)^2 - \frac{1}{2} b\left(\frac{x*x}{6\sqrt{2}} - x_0 x,\ \frac{x*x}{6\sqrt{2}} - x_0 x\right) =$

$$= x_0^4 - x_0^2 b(x,x) + \frac{1}{4} b(x,x)^2 - \frac{1}{144} \underbrace{b(x*x, x*x)}_{= 36|x|^2 \text{ by (B3)}} +$$

$$+ \frac{1}{6\sqrt{2}} \underbrace{b(x*x, x)}_{= 6u(x) \text{ by (B4)}} \cdot x_0 - \frac{1}{2} x_0^2 b(x,x) =$$

$$= x_0^4 - \frac{3}{2} x_0^2 b(x,x) + \frac{1}{\sqrt{2}} u(x) x_0 =$$

$$= x_0 N(\hat{x})$$

(b) $w = \frac{1}{6\sqrt{2}}(z*z) - z_0 z$. We need explicitly $z*z$:

By (B3): $b(x*x, x*x) = 36\, b(x,x)^2$, linearizing $\rightarrow$

$\Rightarrow 4 b(x*x, x*y) = 36 \cdot 4 b(x,x) b(x,y)$ $\rightarrow$ by duality $\rightarrow$

$$\Rightarrow (x*x)*x = 36\, b(x,x) x \qquad \text{(B5)}$$

$$\Rightarrow \boxed{(x*x)*y + 2(x*y)x = 36\, b(x,x) y + 72\, b(x,y) x}$$

Set $y = x*x$: $(x*x)*(x*x) = -2 \cdot 36\, b(x,x)\, x*x + 36\, b(x,x)\, x*x + 72\, b(x, x*x)\, x$

$$\boxed{(x*x)*(x*x) = 432\, u(x) x - 36\, b(x,x)\, x*x} \qquad \text{(B6)}$$

$$z*z = \left(\frac{x*x}{6\sqrt{2}} - x_0 x\right) * \left(\frac{x*x}{6\sqrt{2}} - x_0 x\right) =$$

$$= \frac{1}{72}(x*x)*(x*x) - \frac{x_0}{3\sqrt{2}}(x*x)*x + x_0^2 \cdot x*x =$$

$$= /\text{by } (B6),\ (B5)/ = \frac{432}{72} u(x) x - \frac{36}{72} b(x,x)\, x*x -$$

$$- \frac{x_0}{3\sqrt{2}} \cdot 36\, b(x,x)\, x + x_0^2 \cdot x*x =$$

$$= 6u(x)x - \frac{12}{\sqrt{2}} b(x,x)\, x_0 x + \left(x_0^2 - \frac{1}{2} b(x,x)\right) x*x.$$

Setting this into w:

$$w = \frac{1}{6\sqrt{2}} z*z - z_0 z =$$

$$'' \; z_0$$

$$= \frac{1}{\sqrt{2}} u(x) x - b(x,x)\, x_0 x + \cancel{\frac{1}{6\sqrt{2}}\left(x_0^2 - \frac{1}{2} b(x,x)\right) x*x} -$$

$$- \left(x_0^2 - \frac{1}{2} b(x,x)\right)\left(\cancel{\frac{1}{6\sqrt{2}} x*x} - x_0 x\right) =$$

$$= \left(\frac{1}{\sqrt{2}} u(x) - \frac{3}{2} x_0 b(x,x) + x_0^3\right) x = \underline{N(\hat{x})\, x}$$

Thus $(\hat{x}^{\#})^{\#} = N(\hat{x})\,\hat{x} \Rightarrow N(x)$ is Jordan with respect to the base point $e = (1,0)$.

Jordan structure (explicitly)

Applying (2) on page 5, the Jordan product $\bullet$

$$\hat{x} \bullet \hat{y} = \frac{1}{2}\left(\hat{x} \# \hat{y} + 3x_0 \hat{y} + 3 y_0 \hat{x} - \left(6x_0y_0 - 3b(x,y)e\right)\right)$$

Here

$$\hat{x} \# \hat{y} \overset{\text{def}}{=} (\hat{x}+\hat{y})^{\#} - \hat{x}^{\#} - \hat{y}^{\#} = \text{(see the end of p. 9)}$$

$$= \ldots = \left(2x_0y_0 - b(x,y), \frac{1}{3\sqrt{2}} x * y - x_0 y - y_0 x\right)$$

Hence we obtain component-wisely:

$$(\hat{x} \bullet \hat{y})_0 = \frac{1}{2}\left(2x_0y_0 - b(x,y) + 6x_0y_0 - 6x_0y_0 + 3b(x,y)\right)$$

$$= b(x,y) + x_0y_0 = \frac{1}{3} T(x,y)$$

$$(\hat{x} \bullet \hat{y})_V = \frac{1}{2}\left(\frac{1}{3\sqrt{2}} x * y - x_0 y - y_0 x + 3x_0 y + 3y_0 x\right) =$$

$$= \left(x_0 y + y_0 x + \frac{1}{6\sqrt{2}} x * y\right)$$

Therefor

$$\boxed{\hat{x} \bullet \hat{y} = \left(x_0y_0 + b(x,y), \; x_0 y + y_0 x + \frac{1}{6\sqrt{2}} x * y\right)}$$

is the "hidden" Jordan algebra structure.

$(= J(N,e))$.

Example. $u = x_1^3 - 3x_1x_2^2$ on $\mathbb{F}^2 = V$.

$$\nabla u = (3(x_1^2 - x_2^2), -6x_1x_2)$$

$$|\nabla u|^2 = 9(x_1^2 + x_2^2)^2 = 9\, b(x,x), \quad b(x,y) = x_1y_1 + x_2y_2$$

Now introduce

- $x * y = \nabla u(x+y) - \nabla u(x) - \nabla u(y) = 6(x_1y_1 - x_2y_2, -x_1y_2 - x_2y_1)$
  $= 6\,\bar{x}\cdot\bar{y}$, where "$\cdot$" is complex number multiplication and $(\bar{\cdot})$ is bar.

  $\hat{\mathbb{C}}$ = "paracomplex" numbers: $i \cdot i = 1$

- Note that

  $(x * x) * x = 36\, x\bar{x}x = 36\,|x|^2 x$ (eiconal ident)

- Jordan structure: $\xi = (a, x)$, $\eta = (b, y) \in \mathbb{F} \times \mathbb{F}^2$

$$\boxed{\xi \bullet \eta = \left(ab + \tfrac{1}{2}(x\bar{y} + \bar{x}y), \ ay + bx + \tfrac{1}{\sqrt{2}}\bar{x}\bar{y}\right)}$$

$(\mathbb{F} \times \mathbb{F}^2)$ with $\bullet$ is a 3D - Jordan algebra

Appendix: A) Hsiang algebras

- motivation: Hsiang '67, zero mean curvature hypercones given by $u(x)=0$ in $\mathbb{R}^n$, $u(x)$ - cubic homogeneous polynomial.
- a straigtforward approach does not work even for dim $=3$.
- key idea: given a form $u(x)$, define a metrized algebra $(\mathbb{R}^n, \circ)$ such that

$$x\circ y = \nabla u(x+y) - \nabla u(x) - \nabla u(y)$$

Then

1) $x\circ y = y\circ x$

2) $x\circ x = 4\nabla u(x) - 2\nabla u(x) = 2\nabla u(x)$

$$\nabla u(x) = \frac{x\circ x}{2}, \quad u(x) = \frac{1}{3}\langle \nabla u, x\rangle = \frac{1}{6}\langle x^2, x\rangle$$

3) $\mathrm{Hess}\, u(x) = L(x)$

4) Zero mean curvature equation

$$\Delta u(x)\cdot |\nabla u|^2 - \frac{1}{2}\nabla u^t \cdot \mathrm{Hess}\, u \cdot \nabla u = Q(x)\, u(x)$$

becomes

$$\mathrm{tr}\, L(x) \frac{\langle x_1^2, x^2\rangle}{4} - \frac{1}{4}\langle x^2, x^3\rangle = \frac{1}{6} Q(x) \langle x^2, x\rangle$$

- The case $Q(x) = K\langle x,x\rangle$ is principal.
- In this case one can prove that $\operatorname{tr} L(x) = 0$
- This gives

$$\langle x^2, x^3\rangle = \alpha\langle x^2, x\rangle\langle x, x\rangle$$

Polarization yields

$$2\langle xy, x^3\rangle + \langle x^2, x^2y + 2x(xy)\rangle = 2\alpha\langle x,y\rangle\langle x^2, x\rangle + 3\alpha\langle x^2, y\rangle\langle x, x\rangle$$

$$4x^3x + x^2x^2 - 2\alpha\langle x^2, x\rangle x - 3\alpha\langle x,x\rangle x^2 = 0.$$

Remark. If $(A, \langle,\rangle)$ is metrized and

$$Ax^3x + Bx^2x^2 + C\langle x,x\rangle x^2 + D\langle x^2, x\rangle x = 0$$

then pairing gives

$$A\langle x^3, x^2\rangle + B\langle x^2, x^3\rangle + \langle x,x\rangle\langle x^2, x\rangle(C+D) = 0$$

$\Rightarrow$ Algebra is determined by $(A+B) : (C+D)$.

- $A+B = C+D = 0$
- if not, $\langle x^2, x^3\rangle = \alpha\langle x,x\rangle\langle x^2, x\rangle \rightarrow$ Hsiang

A2) Properties of idempotents in metrized algebras over $\mathbb{R}$ (include all invariant algebra of sporadic finite simple groups.)

general! case

1) $\mathrm{Idm}(A) \neq \emptyset$

Proof $\langle x^2, x\rangle$ attains max on $\langle x, x\rangle = 1$, hence

$\nabla\langle x^2, x\rangle = 3x^2 \parallel \nabla\langle x, x\rangle = x \Rightarrow 3x^2 = \alpha x$ and

$\langle x^2, x\rangle = \frac{\alpha}{3}\langle x, x\rangle = \frac{\alpha}{3} \neq 0$ (max!) $\Rightarrow e = \frac{3}{\alpha}x \in \mathrm{Idm}(A)$ $\blacksquare$

2) "Minimal" idempotents. Note $\forall\, c \in \mathrm{Idm}(A)$:

$\langle c, c\rangle = \langle c^2, c\rangle = u(c)$, $u(\tilde{c}) = \left\langle \left(\frac{c}{|c|}\right)^2, \frac{c}{|c|}\right\rangle = \frac{1}{|c|}$

$\Rightarrow$ the idempotents with $\max\limits_{|x|=1} u(x)$ are exactly "minimal" idempotents, i.e. having the minimal length. They corresponds to primitive idmts: if $c = c_1 + c_2$, $c_1 c_2 = 0$, $c_i^2 = c_i$ then $c^2 = c$ and $\langle c, c\rangle = \langle c_1, c_1\rangle + \langle c_2, c_2\rangle$

Hence "minimal" idempotents are indecomosable. (= primitive)

3) Spectrum. Suppose $c \in \mathrm{Idm}^*(A)$ (=minimal)

Then the spectrum of $L(c) = \{\lambda\}$: either/or:

- $\lambda = 1$ corresponds to c
- $\lambda \leq \frac{1}{2}$

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Proof let $ct = \lambda t$, where $\langle t, c\rangle = 0$, $\langle t,t\rangle = 1$,

$x := (|c|^2+\alpha^2)^{-1/2}(c+\alpha t)$, then $u(x,x) \leq \frac{u(c,c)}{|c|^3}$.

$$x^2 = (|c|^2+\alpha^2)^{-1}(c+2\alpha ct+\alpha^2t^2) = (|c|^2+\alpha^2)^{-1}(c+2\alpha\lambda t+\alpha^2t^2)$$

$$\langle x^2, x\rangle = (|c|^2+\alpha^2)^{-\frac{3}{2}}\langle c+2\alpha t\lambda+\alpha^2t^2; c+\alpha t\rangle =$$

$$= (|c|^2+\alpha^2)^{-\frac{3}{2}}\left(\langle c,c\rangle + 2\lambda\alpha^2\langle t,t\rangle + \alpha^2\langle t^2,c\rangle + \alpha^3\langle t,t^2\rangle\right)$$

$$\langle t^2,c\rangle = \langle t,tc\rangle = \lambda\langle t,t\rangle$$

$$= (|c|^2+\alpha^2)^{-\frac{3}{2}}\left(\langle c,c\rangle + 3\lambda\alpha^2 + \alpha^3\langle t,t^2\rangle\right) \leq \frac{1}{|c|}$$

$$|c|^2 + 3\lambda\alpha^2 + \alpha^3\langle t,t^2\rangle \leq \left(1+\frac{\alpha^2}{|c|^2}\right)^{3/2}|c|^2 =$$

$$= \left(1+\frac{3}{2|c|^2}\alpha^2+\ldots\right)|c|^2$$

$$\Rightarrow \boxed{\lambda \leq \tfrac{1}{2}}$$

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4 In fact if c is "minimal", $\frac{1}{2}\in\sigma(c)$,

then

$$A_c(\tfrac{1}{2})A_c(\lambda) \perp A_c(\lambda) \quad \forall\lambda$$