TSDT14 Signal Theory

Solutions to the exam 2017-10-19

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1

At least two of the following three sub-tasks have to be treated correctly as partial fulfillment to pass the exam:

a. A time-discrete process $X[n]$ is said to be stationary in the wide sense if its ACF $r_X[n_1, n_2]$ only depends on the difference $n_1 - n_2$ and its mean is constant.

b. The PSD $R_Y(f)$ of the output is given by

$$R_Y(f) = |H(f)|^2 R_X(f) = \begin{cases} 9R_0, & 1 < |f|, \\ 0, & \text{elsewhere} \end{cases}$$

c. According to T&F, Page 15, we have the ACF

$$r_Y(\tau) = \frac{2}{\pi} \arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right) = \frac{2}{\pi} \arcsin\big(\operatorname{sinc}(\tau)\big),$$

since the input is Gaussian with mean 0.

2

We are given the ergodic process $X(t)$ with ACF

$$r_X(\tau) = 1 + \operatorname{sinc}(\tau).$$

Since the process is ergodic, we have

$$m_X^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} r_X(\tau)\, d\tau = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 1 + \operatorname{sinc}(\tau)\, d\tau = 1.$$

Thus, we have the two possibilities $m_X = \pm 1$. The PSD of $X(t)$ is given by

$$R_X(f) = \mathcal{F}\big\{r_X(\tau)\big\} = \mathcal{F}\{1 + \operatorname{sinc}(\tau)\} = \delta(f) + \operatorname{rect}(f),$$

according to T&F, Pages 18-19. This signal is the input to an LTI system with impulse response

$$h(t) = e^{-t}u(t).$$

The frequency response of that filter is

$$H(f) = \mathcal{F}\big\{h(t)\big\} = \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1 + j2\pi f},$$

according to T&F, Page 18. The super formula on Page 9 in T&F then gives the PSD of the output $Y(t)$ as

$$R_Y(f) = \big|H(f)\big| R_X(f) = \frac{1}{1 + (2\pi f)^2}\big(\delta(f) + \operatorname{rect}(f)\big) = \delta(f) + \frac{1}{1 + (2\pi f)^2} \operatorname{rect}(f).$$

Its mean is given by

$$m_Y = H(0) m_X = \pm 1.$$

3

Apart from the sampling, that adds nothing significant to the problem, this is an example from one of the lectures.

We are to determine the ACF of the output expressed in the ACF of the input.

$$\begin{aligned} r_Y[k] &= \mathrm{E}\{Y[n]Y[n+k]\} = \\ &= \mathrm{E}\big\{\big(X(nT) + \epsilon X^2(nT)\big)\big(X((n+k)T) + \epsilon X^2((n+k)T)\big)\big\} \\ &= \mathrm{E}\{X(nT)X((n+k)T)\} + \epsilon^2 \mathrm{E}\{X^2(nT)X^2((n+k)T)\} \\ &\quad + \epsilon\left[\mathrm{E}\{X^2(nT)X((n+k)T)\} + \mathrm{E}\{X(nT)X^2((n+k)T)\}\right]. \end{aligned}$$

Here we can identify

$$\mathrm{E}\{X(nT)X((n+k)T)\} = r_X(kT),$$
$$\mathrm{E}\{X^2(nT)X^2((n+k)T)\} = r_X^2(0) + 2r_X^2(kT),$$

using the definition of ACF in T&F, Page 7, and using the expression for squaring a Gaussian process with mean zero in T&F, Page 15. What is left now are the two mixed terms. On Page 16 in T&F, we have the expression

$$\begin{aligned} \mathrm{E}\{ABC\} &= \\ &= \mathrm{E}\{AB\}\mathrm{E}\{C\} + \mathrm{E}\{AC\}\mathrm{E}\{B\} + \mathrm{E}\{A\}\mathrm{E}\{BC\} \\ &\quad - 2\mathrm{E}\{A\}\mathrm{E}\{B\}\mathrm{E}\{C\} \end{aligned}$$

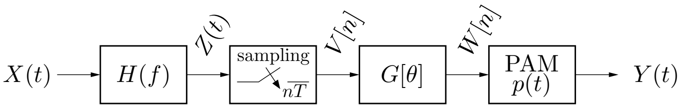
that holds if (A, B, C) are jointly Gaussian. With $A = B = X(nT)$ and $C = X((n+k)T)$, we get $\mathrm{E}\{X^2(nT)X((n+k)T)\} = 0$ since we have $m_X = 0$. Similarily, we have $\mathrm{E}\{X(nT)X^2((n+k)T)\} = 0$. Totally, we have

$$r_Y[k] = r_X(kT) + \epsilon^2 \left(r_X^2(0) + 2r_X^2(kT) \right)$$

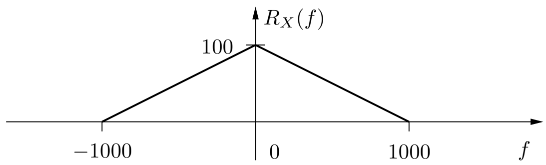
Note: All odd moments of jointly Gaussian variables with mean zero are zero.

4

We denote the various signals according to the figure below.



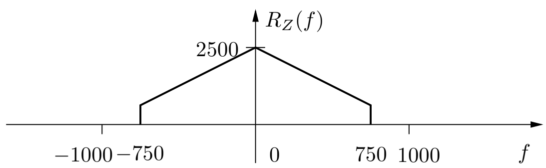
The spectrum of the input as a graph:



For the first filter, $H(f)$, we have the relation

$$R_Z(f) = \left|H(f)\right|^2 R_X(f),$$

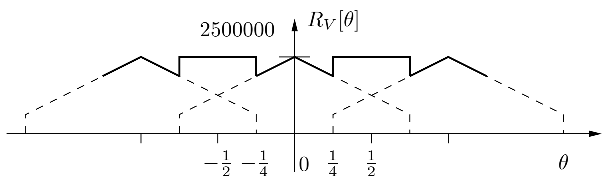
according to T&F, p. 9:



For the sampler, we have the relation

$$R_V[\theta] = f_\mathrm{s} \sum_m R_Z\big(f_s(\theta - m)\big),$$

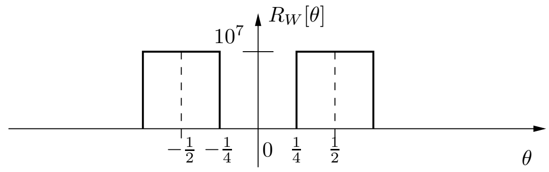
according to T&F, p. 11:



For the second filter, $G[\theta]$, we have the relation

$$R_W[\theta] = \left|G[\theta]\right|^2 R_V[\theta],$$

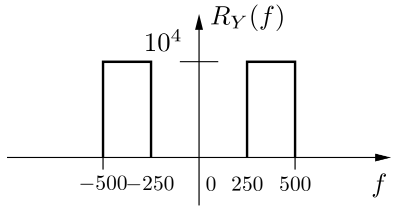
according to T&F, p. 9:



Finally, for the pulse-amplitude modulator, we have the relation

$$R_Y(f) = f_\mathrm{s}\left|P(f)\right|^2 R_W[f/f_\mathrm{s}],$$

according to T&F, p. 12:



The PSD of the output $Y(t)$ can be written as

$$R_Y(f) = \begin{cases} 10^4, & 250 \le |f| \le 500, \\ 0, & \text{elsewhere.} \end{cases}$$

The power of $Y(t)$ is given by

$$P_Y = E\{Y^2(t)\} = r_Y(0) = \int_{-\infty}^{\infty} R_Y(f)df = 5 \cdot 10^6.$$

5

The stochastic variables A and B are uniformly distributed over the interval $[-1, 1]$. Therefore, they have the mean

$$m_A = m_B = 0$$

and variance

$$\sigma_A^2 = \sigma_B^2 = \frac{1}{3}.$$

Since they are uncorrelated, we also have

$$E\{AB\} = m_A m_B = 0.$$

To show that the process is WSS, we need to show that the mean is constant and the the ACF only depends on the difference between the two time instances. For the mean, we have

$$\begin{aligned} m_X(t) &= E\{X(t)\} = E\{A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)\} \\ &= m_A \cos(2\pi f_0 t) + m_B \sin(2\pi f_0 t) \\ &= 0, \end{aligned}$$

where we have used the observations above and the linearity of the expectation. Obviously this mean is independent of t. For the ACF, we have

$$\begin{aligned} & r_X(t, t+\tau) = \\ &= E\{X(t)X(t+\tau)\} \\ &= E\{A^2\}\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) \\ &+ E\{AB\}\cos(2\pi f_0 t)\sin(2\pi f_0(t+\tau)) \\ &+ E\{BA\}\sin(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) \\ &+ E\{B^2\}\sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau)) \\ &= \frac{1}{3}\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) \\ &+ \frac{1}{3}\sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau)) \\ &= \frac{1}{3}\cos(2\pi f_0 \tau) = r_X(\tau), \end{aligned}$$

which is independent of t. Thus, the process is WSS.

6

We are given the two space-continuous signals

$$x(a_1, a_2) = \begin{cases} 1, & |a_1| < \frac{1}{2}, |a_2| < 1, \\ 0, & \text{elsewhere}, \end{cases}$$

$$y(a_1, a_2) = \begin{cases} 1, & |a_1| < 1, |a_2| < \frac{1}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

Those are obviously separable as

$$\begin{aligned} x(a_1, a_2) &= z_1(a_1) \cdot z_2(a_2), \\ y(a_1, a_2) &= z_2(a_1) \cdot z_1(a_2), \end{aligned}$$

where we have

$$z_1(a) = \text{rect}(a), \qquad z_2(a) = \text{rect}(a/2).$$

a. The 2-D convolution $(x \circledast y)(a_1, a_2)$ is therefore separable as

$$(x \circledast y)(a_1, a_2) = (z_1 * z_2)(a_1) \cdot (z_1 * z_2)(a_2).$$

Thus, we need to determine the 1-D convolution

$$(z_1 * z_2)(a) = \begin{cases} 1, & |a| \le \frac{1}{2}, \\ \frac{3}{2} - |a|, & \frac{1}{2} < |a| \le \frac{3}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

Totally, we have

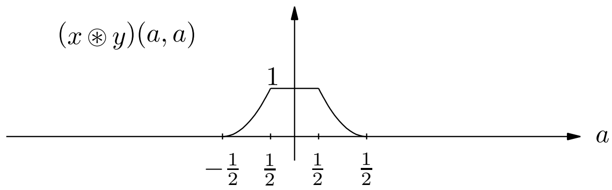
$$(x \circledast y)(a_1, a_2) =$$

$$= \begin{cases} 1, & |a_1| \le \frac{1}{2} \ \& \ |a_2| \le \frac{1}{2} \\ \frac{3}{2} - |a_1|, & \frac{1}{2} < |a_1| \le \frac{3}{2} \ \& \ |a_2| \le \frac{1}{2}, \\ \frac{3}{2} - |a_2|, & |a_1| \le \frac{1}{2} \ \& \ \frac{1}{2} < |a_2| \le \frac{3}{2}, \\ \left(\frac{3}{2} - |a_1|\right)\left(\frac{3}{2} - |a_2|\right), & \frac{1}{2} < |a_1| \le \frac{3}{2} \ \& \ \frac{1}{2} < |a_2| \le \frac{3}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

b. Plugging in $a_1 = a_2 = a$ in the expression above, we get

$$(x \circledast y)(a, a) = \begin{cases} 1, & |a| \le \frac{1}{2} \\ \left(\frac{3}{2} - |a|\right)^2, & \frac{1}{2} < |a| \le \frac{3}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

Graphically:



Note:
We have

$$z_2(a) = \text{rect}(a + 1/2) + \text{rect}(a - 1/2).$$

Recall

$$(\text{rect} * \text{rect})(a) = \text{triangle}(a).$$

Then we can write

$$\begin{aligned} & (z_1 * z_2)(a) = \\ & = \text{triangle}(a + 1/2) + \text{triangle}(a - 1/2), \end{aligned}$$

$$\begin{aligned} & (x \circledast y)(a_1, a_2) = \\ & = \big(\text{triangle}(a_1 + 1/2) + \text{triangle}(a_1 - 1/2)\big) \cdot \\ & \quad \cdot \big(\text{triangle}(a_2 + 1/2) + \text{triangle}(a_2 - 1/2)\big), \end{aligned}$$

$$\begin{aligned} & (x \circledast y)(a, a) = \\ & = \big(\text{triangle}(a + 1/2) + \text{triangle}(a - 1/2)\big)^2. \end{aligned}$$