

# TSDT14 Signal Theory

## Lecture 5

### Prediction and Non-Linear Systems

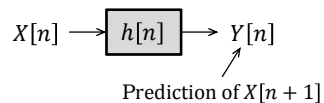
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## Prediction

WSS TD process  $X[n]$ .

Task: Predict next sample.

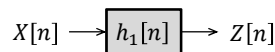
Method: Filter the signal.



Prediction error:

$$Z[n] = Y[n] - X[n+1].$$

Interpretation:



Prediction error filter:

$$h_1[n] = h[n] - \delta[n+1]$$

$$H_1[\theta] = H[\theta] - e^{j2\pi\theta}$$

Mean:  $m_Z = (H[0] - 1)m_X$

Unbiased:  $m_Z = 0$

$$\Rightarrow H[0] = 1 \text{ or } m_X = 0$$

Quality measure (distorsion):

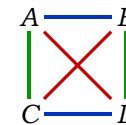
$$\varepsilon^2 = r_Z[0] = (h_1 * \tilde{h}_1 * r_X)[0]$$

$$= \int_0^1 |H_1[\theta]|^2 R_X[\theta] d\theta$$

$$\tilde{h}_1[n] = h_1[-n]$$

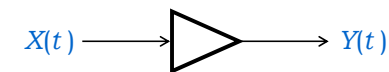
## 3- & 4-Dim. Jointly Gaussian Variables

$$E\{ABCD\} = E\{AB\} \cdot E\{CD\} + E\{AC\} \cdot E\{BD\} + E\{AD\} \cdot E\{BC\} - 2 \cdot E\{A\} \cdot E\{B\} \cdot E\{C\} \cdot E\{D\}$$



$$E\{ABC\} = E\{AB\} \cdot E\{C\} + E\{AC\} \cdot E\{B\} + E\{A\} \cdot E\{BC\} - 2 \cdot E\{A\} \cdot E\{B\} \cdot E\{C\}$$

## Example: Amplifier 1(3)



Input:  $X(t)$  stationary Gaussian process with  $m_X = 0$ .

Output:  $Y(t) = X(t) + \varepsilon X^2(t)$ . Quadratic distorsion.

Mean:  $m_Y = E\{X(t) + \varepsilon X^2(t)\} = \varepsilon \cdot E\{X^2(t)\} = \varepsilon \cdot r_X(0)$

$$m_X = 0$$

$$\begin{aligned} \text{ACF: } r_Y(\tau) &= E\{Y(t)Y(t+\tau)\} = E\{(X(t) + \varepsilon X^2(t))(X(t+\tau) + \varepsilon X^2(t+\tau))\} \\ &= \underbrace{E\{X(t)X(t+\tau)\}}_{r_X(\tau)} + \varepsilon \cdot \underbrace{E\{X^2(t)X(t+\tau)\}}_{=?} + \varepsilon \cdot \underbrace{E\{X(t)X^2(t+\tau)\}}_{=?} + \varepsilon^2 \cdot \underbrace{E\{X^2(t)X^2(t+\tau)\}}_{=?} \end{aligned}$$

## Example: Amplifier 2(3)

$$\begin{aligned} E\{X^2(t)X(t+\tau)\} &= \\ &= E\{X^2(t)\} \cdot E\{X(t+\tau)\} + 2E\{X(t)\} \cdot E\{X(t)X(t+\tau)\} - 2E^2\{X(t)\} \cdot E\{X(t+\tau)\} \\ &= r_X(0) \cdot m_X + 2 \cdot m_X \cdot r_X(\tau) - 2 \cdot m_X^3 = m_X \cdot (r_X(0) + 2r_X(\tau) - 2m_X^2) = 0 \end{aligned}$$

$$E\{X(t)X^2(t+\tau)\} = \dots = m_X \cdot (r_X(0) + 2r_X(\tau) - 2m_X^2) = 0 \quad \leftarrow \text{Since } m_X=0$$

$$\begin{aligned} E\{X^2(t)X^2(t+\tau)\} &= \\ &= E\{X^2(t)\} \cdot E\{X^2(t+\tau)\} + 2E^2\{X(t)X(t+\tau)\} - 2E^2\{X(t)\} \cdot E^2\{X(t+\tau)\} \\ &= r_X^2(0) + 2 \cdot r_X^2(\tau) - 2 \cdot m_X^4 = r_X^2(0) + 2 \cdot r_X^2(\tau) \quad \leftarrow \text{Since } m_X=0 \end{aligned}$$

$$\begin{aligned} \text{Totally: } r_Y(\tau) &= r_X(\tau) + \varepsilon^2 \cdot (r_X^2(0) + 2 \cdot r_X^2(\tau)) \\ R_Y(f) &= R_X(f) + \varepsilon^2 \cdot (r_X^2(0) \cdot \delta(f) + 2 \cdot (R_X * R_X)(f)) \end{aligned}$$

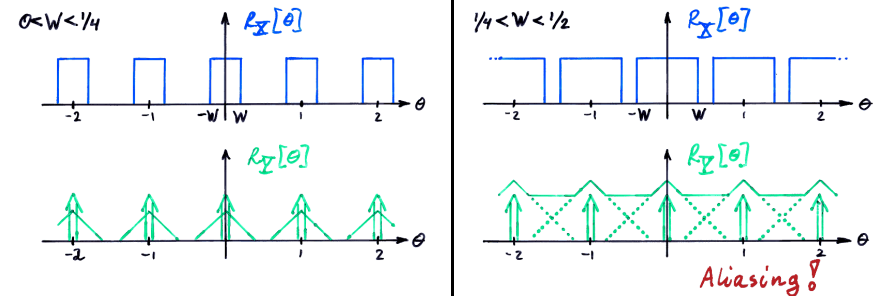
## Squaring a Time-Discrete Gaussian Process

Input:  $X[n]$  stationary Gaussian with  $m_X=0$ . (Bandlimited)

$$\text{Output: } Y[n] = X^2[n] \quad r_Y[k] = r_X^2[0] + 2r_X^2[k]$$

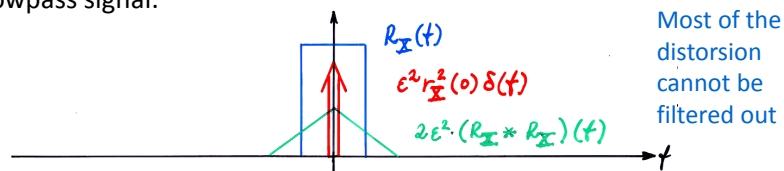
$$R_Y[\theta] = r_X^2[0] \cdot \sum_m \delta(\theta - m) + 2(R_X \otimes R_X)[\theta]$$

$$\text{Periodic convolution: } (R_X \otimes R_X)[\theta] = \int_0^1 R_X[\phi] R_X[\theta - \phi] d\phi$$

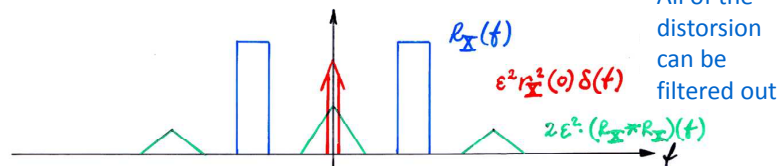


## Example: Amplifier 3(3)

Lowpass signal:



Bandpass signal:



## Price's Theorem

Useful to determine the ACF after a nonlinear mapping of Gaussian processes.

Prerequisites:  $(A, B)$  is jointly Gaussian with mean  $(0, 0)$ ,  $f(a)$  and  $g(b)$  are functions, usually nonlinear, and  $\rho = E\{AB\}$ .

$$\text{Then we have: } \frac{\partial^n}{\partial \rho^n} E\{f(A)g(B)\} = E\left\{ \frac{\partial^n}{\partial A^n} f(A) \frac{\partial^n}{\partial B^n} g(B) \right\}$$

## Example Price's Theorem 1(2)

Assumptions:  $\mathbf{X}(t)$  strictly stationary Gaussian Process with  $m_{\mathbf{X}}=0$ .  
 $\mathbf{Y}(t) = \mathbf{X}^2(t)$

Objective: Express  $r_{\mathbf{Y}}(\tau)$  in  $r_{\mathbf{X}}(\tau)$ .

Method: Price's theorem with  $A=\mathbf{X}(t)$ ,  $B=\mathbf{X}(t+\tau)$ ,  
 $f(a) = g(a) = a^2$  and  $S = r_{\mathbf{X}}(\tau)$ .

We have  $r_{\mathbf{Y}}(\tau) = E\{\mathbf{Y}(t)\mathbf{Y}(t+\tau)\} = E\{\mathbf{X}^2(t)\mathbf{X}^2(t+\tau)\} = E\{f(\mathbf{X}(t))\cdot g(\mathbf{X}(t+\tau))\}$

Price's theorem for  $n=1$  gives us

$$\frac{\partial}{\partial r_{\mathbf{X}}(\tau)} r_{\mathbf{Y}}(\tau) = E\left\{\frac{\partial}{\partial A} f(A) \cdot \frac{\partial}{\partial B} g(B)\right\} = E\left\{\frac{\partial A^2}{\partial A} \cdot \frac{\partial B^2}{\partial B}\right\} = E\{2A \cdot 2B\}$$

$$= 4 E\{\mathbf{X}(t)\mathbf{X}(t+\tau)\} = 4 r_{\mathbf{X}}(\tau)$$

This gives us

$$\partial r_{\mathbf{Y}}(\tau) = 4 r_{\mathbf{X}}(\tau) \cdot \partial r_{\mathbf{X}}(\tau) \Rightarrow \int \partial r_{\mathbf{Y}}(\tau) = \int 4 r_{\mathbf{X}}(\tau) \partial r_{\mathbf{X}}(\tau) \Rightarrow$$

$$\Rightarrow r_{\mathbf{Y}}(\tau) + C_1 = 2 r_{\mathbf{X}}^2(\tau) + C_2 \Rightarrow r_{\mathbf{Y}}(\tau) = 2 r_{\mathbf{X}}^2(\tau) + C$$

## Example Price's Theorem 2(2)

We had:  $r_{\mathbf{Y}}(\tau) = 2 r_{\mathbf{X}}^2(\tau) + C$

Objective: Determine the constant  $C$ .

How? Choose a  $\tau$  such that  $r_{\mathbf{X}}(\tau) = 0$  holds.

Then:  $\mathbf{X}(t)$  &  $\mathbf{X}(t+\tau)$  are uncorrelated. }  
 But:  $\mathbf{X}(t)$  &  $\mathbf{X}(t+\tau)$  jointly Gaussian }  $\Rightarrow \mathbf{X}(t)$  &  $\mathbf{X}(t+\tau)$  indep.

So:  $C = r_{\mathbf{Y}}(\tau) = E\{\mathbf{X}^2(t)\mathbf{X}^2(t+\tau)\} = E\{\mathbf{X}^2(t)\} \cdot E\{\mathbf{X}^2(t+\tau)\} = r_{\mathbf{X}}^2(0)$   
 $r_{\mathbf{X}}(\tau) = 0$       Indep.  $r_{\mathbf{X}}(0)$        $r_{\mathbf{X}}(0)$

Totally:  $r_{\mathbf{Y}}(\tau) = 2 r_{\mathbf{X}}^2(\tau) + r_{\mathbf{X}}^2(0)$