Course plan 2018 for the course TMMS11: Models of Mechanics.

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1 General

The course consists of lectures (föreläsningar), classworks (lektioner) and tutorials. The lectures are primarily spent on presentation of the theoretical content; during classworks, solutions to exercises are presented; and during tutorials we discuss solutions handed in by the student the day before the tutorial.

Four home assignments should be presented in written form. These are described in Section 7 and are the following:

- Central force motion for different hypothetical "gravitational" laws.
- Comparison between a Timoshenko beam and an Euler-Bernoulli beam.
- Shallow water equations
- Viscosity measurement formula.

Course requirements are given in more detail below.

Lecture notes can be found at https://modelsofmechanics.wordpress.com/.

2 Teachers

- Anders Klarbring (AK), office 3A:978
- Stefan Lindström (SL), office 3A:987

3 Course material

The course text book

- A Klarbring, *Models of Mechanics*, Springer 2006

can be bought from Bokakademin (Kårallen). It can also be downloaded as an E-book from Linköping University library.

4 Organization

- **Introduction**
  - 2 lectures
  - 1 tutorial
- **Discrete model**
  - 2 lectures
  - 1 exercises
  - 1 tutorial
  - home assignment (computer session)
- **1D model**
  - General
    * 2 lectures
    * 1 exercise
  - Pipe flow
    * 2 lectures
    * 1 exercise
  - Small displ. theory – beams
    * 2 lectures
    * 1 exercise
    * home assignment (computer session)
  - Shallow water equation
    * 1 lecture
    * 2 tutorials
    * computer session
- **3D model**
  - General
    * 1 lecture
  - Small displ. theory – linear elasticity
    * 1 lecture
  - Fluid
    * 1 lecture
    * 1 exercise
    * home assignment

5 Schedule (for the first part of the course)

Each occasion takes 2×45 minutes, and is a lecture (L), a classwork session (E) or a tutorial (T).

Occasion	L/E/T	Contents	Date and time
1	L	Introduction, universal laws I (AK)	3/9 10-12
2	L	Introduction, universal laws II (AK)	6/9 8-10
3	T	Tutorial 1 (AK)	7/9 15-17
4	L	The discrete model I (AK)	10/9 10-12
5	L	The discrete model II (AK)	11/9 15-17
6	E	Exercises in Chapter 4 (SL)	18/9 15-17
7	T	Tutorial 2 (SL)	21/9 15-17
8	L	1D model, general I (AK)	27/9 8-10
9	L	1D model, general II (AK)	2/10 13-157
10	E	Exercises in Chapter 5 (SL)	4/10 8-10
11	L	1D model, pipe flow I (AK)	8/10 10-12
12	L	1D model, pipe flow II (AK)	9/10 13-15
13	T	Tutorial 3, computer session (SL)	9/10 15-17
14	E	Exercises in Chapter 10 (AK)	16/10 15-17
15	T	Tutorial 4 (AK)	18/10 8-10

6 Course requirements

1. It is mandatory to hand in solutions for exercises to be discussed at tutorials, as described in Section 9.

2. Each of the four home assignments are graded by a scale from 1 to 5 points. To pass the course (i.e., to be given at least the grade 3) a minimum of 2 points is required on each assignment. Grade 4 requires a minimum total of 13 points and grade 5 a minimum total of 17 points.

   Each assignment has two deadlines. If the first deadline is meet, the assignment is returned graded with comments in a few days, and a corrected version can be handed in before the second deadline. Results from the first deadline do not affect the final grade.

   Central force motion for different hypothetical "gravitational" laws. First deadline **October 16**. Second deadline **November 5**.

   Comparison between a Timoshenko beam and an Euler-Bernoulli beam. First deadline **?**. Second deadline **?**.

   Shallow water equations. First deadline **?**. Second deadline **?**.

   Viscosity measurement formula. First deadline **?**. Second deadline **?**.

If failing to make the second deadlines, in general, only grade 3 is given, and an additional oral test may be required.

7 Home assignments

7.1 Central force motion for hypothetical "gravitational" laws

Consider two bodies modeled as particles, and affecting each other by an attracting central force

$$f(d) = \frac{k}{d^\alpha},$$

where k and α are real constants with $k > 0$ and $-1 \le \alpha \le 2$, and d is the distance between the two particles. The classical Newton's law of gravitation is recovered as a special case when $\alpha = 2$ and $k = Gm_1m_2$, where G is the universal constant of gravity and m_1 and m_2 are the masses of the two particles.

- Show that the center of mass of the two-particle system has zero acceleration.

- Derive an equation of motion for the relative distance between the particles, i.e., if $\boldsymbol{r}(X_1, t)$ and $\boldsymbol{r}(X_2, t)$ are position vectors for the two particles, derive a second order equation for the relative distance $\boldsymbol{r} = \boldsymbol{r}(X_1, t) - \boldsymbol{r}(X_2, t)$ of the type

$$\gamma \ddot{\boldsymbol{r}} = \boldsymbol{F}, \tag{1}$$

  where expressions for $\gamma > 0$ and $\boldsymbol{F}$ should be given.

- Show that the motion will always take place in one and the same plane of the physical space. (An optional problem that may take some calculation!)

- Introduce a cartesian base and solve (1) for different values of α using MATLAB. Let the values of the other constants, γ and k, be in the order of unity for numerical conditioning. (This is no restriction, cf. the introduction of dimensionless variables in Tutorial 6.) Is there a difference in the general behavior when $\alpha = -1$ and 2 from when $-1 < \alpha < 2$?

- Write a report including a derivation of the governing equations, numerical results with comments and your MATLAB code. It is expected that the report is written using full sentences and with all notations being defined.

- Consult the literature on the two-body central force problem and Bertrand's theorem. Describe in your report how your numerical results relate to Bertrand's theorem.

7.2 Comparison between a Timoshenko beam and an Euler-Bernoulli beam

Consider a beam built in at both ends. The beam is straight with constant cross section area and subjected to a uniformly distributed normal force per unit length, $b_n(s) = \hat{b}_n = const.$. The shear factor k appearing in the Timoshenko beam theory can be identified as $k = \kappa^2 AG$, where κ^2 is a factor depending on the shape of the cross section and Poisson's ratio, A is the cross section area and G is the shear modulus. For a rectangular cross-section and Poisson's ratio $\nu = 0.3$, $\kappa^2 \simeq 0.85$.

Consider the following cases:

1. **Short Beam**

   (a) Solve the Timoshenko beam problem for this situation, using equation-based modeling in COMSOL. Plot[1] the transverse displacement

[1] To open a new Figure window: Postprocessing → Plot Parameters/General/Plot in:/New figure

u_n and the shearing[2] $\varepsilon_n = \partial u_n/\partial x - \varphi$ along the beam. Determine the maximum displacement[3].

(b) Compute the maximum displacement for the corresponding two-dimensional problem using the Structural Mechanics Module (SMM/Plane Stress) in COMSOL.

(c) Compute the maximum displacements for the Timoshenko and Euler-Bernoulli beam using closed-form solutions. (This problem will be addressed in a class work session.)

(d) Investigate what happens if you increase the shear factor in COMSOL. Do you get the result of the Euler-Bernoulli beam? When do you encounter numerical problems?

2. **Long Beam** Repeat **(a)**, **(c)** and **(d)**. Solution of **(b)** for the long beam is not mandatory.

Write a report including a derivation of the governing equations and numerical results with comments. It is expected that the report is written using full sentences and with all notations being defined.

There are two versions of COMSOL Multiphysics available on the computers in the computer lab. Be careful to choose COMSOL version 5.0 in Windows Start Menu!

7.3 Shallow water equations

The flow of water as it approaches a beach is essentially a one-dimensional mechanical motion if we average (neglect) the variation of velocity with depth. The purpose of this assignment is to develop such a one-dimensional model and investigate its behavior by numerical simulation and reasoning based on closed form solutions. It can also be concluded that other water flow situations where the whole water mass from bottom to surface is in motion, such as a tsunami, can be modelled in this way.

Step 1: Governing equations

Starting from general principles, establish the following two governing equations:

(a) For isochoric motion it holds that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial s}((h + h_0)v) = 0, \qquad (2)$$

where

[2] Postprocessing → Plot Parameters/Line/Expression. Here, a new variable can be defined. $\partial u_n/\partial x = ux$ etc.

[3] Postprocessing → Plot Parameters/General/Select Max/min marker

- s is the horizontal coordinate and t denotes time;
- $h = h(s,t)$ measures the water level deviation from a certain reference level;
- $h_0 = h_0(s)$ gives the water depth (bottom to reference level);
- $v = v(s,t)$ is the velocity of the water.

Figure 1 illustrates the situation.

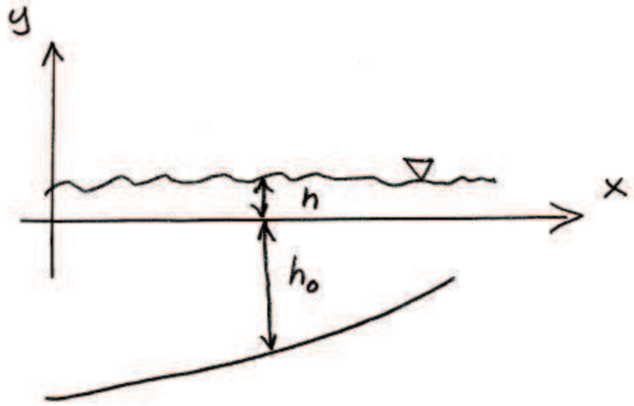

Figure 1: The geometry of shallow water.

(b) The equation of motion (Euler I) results in

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} + g\frac{\partial h}{\partial s} = 0, \tag{3}$$

where g is the acceleration of gravitation.

Equation (3) can be derived by generalizing the approach of Section 6.6 in the textbook. The domain $\hat{\mathcal{P}}_t$ is illustrated in Figure 2. We substitute the following two terms into equation (6.16) of the textbook:

$$\boldsymbol{q}_V = -\rho g \boldsymbol{e}_y, \tag{4}$$

$$p = p_* + \rho g(h(x,t) - y), \tag{5}$$

where ρ is the density, p_* is the atmospheric pressure at the water level, y is the vertical coordinate and $\boldsymbol{e}_y$ is the upward unit vector. Equation (5) is known as the hydrostatic assumption. It can be shown to be exactly valid in the present case due to no vertical velocity.

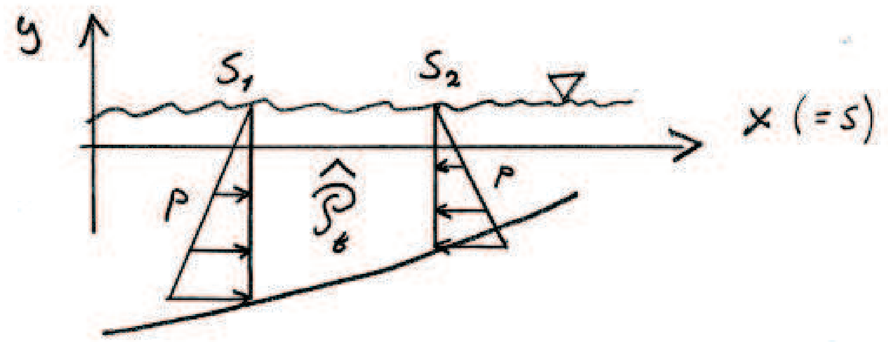

Figure 2: The domain $\hat{\mathcal{P}}_t$ and the hydrostatic assumption.

Step 2: Linearization and wave speed

Equations (2) and (3) constitute, together with appropriate boundary conditions and data in the form of g and h_0, a mathematical problem for the unknown functions $h(s,t)$ and $v(s,t)$. In fact, it is a nonlinear wave propagation problem that can be written in a simultaneous form as

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ v \end{pmatrix} = -\begin{pmatrix} v & h+h_0 \\ g & v \end{pmatrix}\frac{\partial}{\partial s}\begin{pmatrix} h \\ v \end{pmatrix} - \begin{pmatrix} \frac{\partial h_0}{\partial s} v \\ 0 \end{pmatrix}. \tag{6}$$

Linearize (6) at $v = 0$ and $h = 0$, and find an expression for the wave speed. Here you may wish to compare to reports on times when different countries were hit by the 2004 tsunami (or some other tsunami for which there is data available).

Step 3: Shore waves

Solve (6) and its linearized counterpart by means of COMSOL for some appropriate depth profiles h_0 and boundary conditions. You may wish to use dimensionless variables. Compare the two numerical solutions and compare also to D'Alambert's analytical solution. Suggested boundary conditions may be

- At deep water we may assume $h = \sin \omega t$ at a certain location x_{left} and for some frequency ω. This can be a boundary condition for a shallow water equation valid between x_{left} and the shore at x_{right}. What boundary conditions may be assumed at x_{right}?

Step 4: Tsunami

To simulate a tsunami one could attempt to let h_0 depend on time and place in some appropriate way to simulate an earthquake at the bottom of the see.

What modifications of the equations need to be done for this? Make some simulations!

7.4 Viscosity measurement formula

Consider a Newtonian fluid confined to the space between two concentric cylinders as in Figure 3. The inner cylinder moves axially with a constant velocity ν relative to the outer cylinder. A way of measuring the viscosity of the fluid is to measure the external axial force required for maintaining the relative velocity. The exercise involves deriving a formula for this axial force by first deriving the velocity field of the fluid. Rotational symmetry as well as laminar flow, meaning that only the axial velocity component of the fluid is non-zero, may be assumed.

Clearly cylindrical coordinates are favorably used, so a first step is to establish the relevant equations in a cylindrical frame. The details of this step need not be detailed in the report, while other steps in the derivation should be carefully described. Again, it is expected that the report is written using full sentences and with all notations being defined.

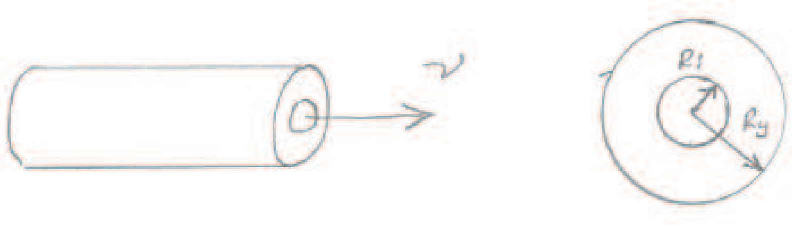

Figure 3: Two concentric cylinders in axial movement.

8 General comments on reports

The reports should be in the form of proper scientific papers, i.e.,

- **All** notations used need to be defined. The only exceptions may be things like $\pi \approx 3.14$.
- Assumptions (or simplifications) should be clearly stated, and you should distinguish between mathematical assumptions and assumptions of physical origin. These assumptions should be part of a coherent deduction ending in the complete problems being solved and investigated.
- The language should **not** be the type of shorthand language that a teacher may use on a black (white) board. Grammatically correct full sentences should be used.

- There should be a conclusion section that can be read independently of the rest of the report.

A general comment on the level of reports is that they should be readable by a student with your own background that has not taken the course Models of Mechanics.

Finally, reports should be written and worked out independently by each student. Obviously, participants of the course can (and should) communicate and discuss the problems treated, but lack of individual achievement and understanding is usually easy to spot.

9 Tutorials

Handwritten solutions for tutorials should, with some exceptions, have arrived outside the teachers office 12.00 the day before the tutorial.

Tutorial 1

As a preparation for this tutorial, hand in solutions for

- Exercise 1: Write down a **complete model** that you know from previous studies. Preferably it should be a differential equation that one solves to get an answer. What is the data and what is the answer? What about boundary or initial conditions? "M:are" may think of bar and beam equations and linear elasticity. "Y:are" may want to think of the heat flow equation or the wave equation.

- Exercise 2: An equation from mechanics can be

  1. A fundamental equation[4].
  2. A particular law[5].
  3. Something formed by combining fundamental laws and particular laws.
  4. Simply a mathematical definition, which is not really an equation in the sense that it includes no new physical content.

[4] Fundamental (or universal, or general) equations are those that are based solely on the three universal principles:

(a) "Kraftbalans"

(b) "Momentbalans"

(c) "Massans bevarande"

[5] Particular laws are those that tells us how the forces work in different physical situations, e.g., different particular laws are needed to distinguish between a solid body and a fluid body. Particular laws can also be said to be everything extra besides fundamental equations that are needed to form a complete problem.

Look up some equations that you have been told about in previous courses and see if you can make out which of the above categories they belong to. Some suggestions may be the equation defining a Newtonian fluid, Newton's law of gravitation, the differential equation for bending of a beam, the law of friction forces, the continuity equation, etc.

Tutorial 2

As a preparation for this tutorial, hand in solutions for

- **Exercise 4.1**
- **Exercise 4.9:** (Use cartesian coordinates.)

Tutorial 3 (Computer session)

The third tutorial concerns the home assignment "Central force motion for hypothetical gravitational laws". Prepare the following, that you do not have to hand in before the session:

- Formulate the equations of motion (Two 2'nd order ODE's). (Use cartesian coordinates.)
- Rewrite these as a system of 1'st order ODE's.
- Make sure you are well acquainted with the ODE-solver in MATLAB. Consult for instance a MATLAB user guide or the brief introduction by P.W. Christensen (handed out on request).

Tutorial 4

Hand in solutions for the following:

- Prove that the following equations are correct:

$$\frac{\partial \boldsymbol{e}_t(s)}{\partial s} = \kappa(s)\boldsymbol{e}_n(s) \tag{7}$$

  and

$$\frac{\partial \boldsymbol{e}_n(s)}{\partial s} = -\kappa(s)\boldsymbol{e}_t(s). \tag{8}$$

  You may follow the derivation in the textbook.

- Make a derivation of (prove)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}((h+h_0)v) = 0, \tag{9}$$

  which is one of the equations of the assignment on the shallow flow equations. Be sure to list assumptions leading to this equation.

- Make a derivation of (prove)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0, \tag{10}$$

which is one of the equations of the assignment on the shallow flow equations. Be sure to list assumptions leading to this equation.

Tutorial 5 (Computer session)

This tutorial concerns the home assignment "Comparison between a Timoshenko beam and an Euler-Bernoulli beam".

Tutorial 6

The full nonlinear shallow water flow problem (eqs. (2) and (3)) reads

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ v \end{pmatrix} = -\begin{pmatrix} v & h+h_0 \\ g & v \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} h \\ v \end{pmatrix} - \begin{pmatrix} \frac{\partial h_0}{\partial x} v \\ 0 \end{pmatrix}. \tag{11}$$

Linearizing around $v = h = 0$ gives

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ v \end{pmatrix} = -\begin{pmatrix} 0 & h_0 \\ g & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} h \\ v \end{pmatrix}. \tag{12}$$

- Solve (12) analytically for h_0 =constant. That is, derive D'Alamberts solution. Identify from this solution the wave speed and compare to reports of the recent tsunami.

- Introduction of dimensionless variables has many advantages. (1) Numerical computations become better conditioned. (2) You solve the problem in one shot for a whole family of data. Introduce the following dimensionless variables

$$\bar{x} = \frac{x}{\ell}, \quad \bar{t} = \frac{t\upsilon}{\ell}, \quad \bar{v} = \frac{v}{\upsilon}, \quad \bar{h} = \frac{hg}{\upsilon^2}, \quad \bar{h}_0 = \frac{h_0 g}{\upsilon^2},$$

where ℓ and υ are numbers with dimension length and velocity, respectively. Rewrite (11) in dimensionless form. Note how, when h_0 =constant, it is favorable to let $\upsilon = \sqrt{gh_0}$.

Tutorial 7 (Computer session)

This tutorial concerns the home assignment "Shallow water equations". Nothing needs to be handed in beforehand, but you should be will prepared.