These subtasks have to be treated correctly as partial fulfillment to pass the exam.

a. A stochastic process is said to be exactly predictable if it is enough to observe a realization during a finite interval to determine the complete realization.

b. The PSD $R_Y[\theta]$ is given by Poisson’s summation formula for stochastic processes (T & F p. 11):

$$R_Y[\theta] = f_s \sum_m R_X(f_s(\theta - m))$$

Graphically:

![Graphical representation of $R_Y[\theta]$](image)

For the input, we have the PDF

$$f_{X[n]}(x) = \frac{1}{2} \text{rect} \left( \frac{x}{2} \right).$$

and the mean

$$m_X = \int_{-\infty}^{\infty} x f_{X[n]}(x) \, dx = \int_{-1}^{1} \frac{x}{2} \, dx = 0.$$  

To determine the ACF for the input, we study the cases $k = 0$ and $k \neq 0$ separately. For $k = 0$ we get

$$r_X[0] = \mathbb{E}\{X^2[n]\} = \int_{-\infty}^{\infty} x^2 f_{X[n]}(x) \, dx = \int_{-1}^{1} \frac{x^2}{2} \, dx = \frac{1}{3},$$

and for $k \neq 0$ we have

$$r_X[k] = \mathbb{E}\{X[n+k]X[n]\} = \mathbb{E}\{X[n+k]\} \mathbb{E}\{X[n]\} = m_X^2 = 0,$$

where we have used the independence in the second equality. Totally, we have

$$r_X[k] = \frac{1}{3} \delta[k].$$

a. To determine the ACF $r_Y[k]$, we notice from the figure that we have

$$Y[n] = X[n] + X[n-1].$$
The ACF is then given by

\[ r_Y[k] = E\{Y[n+k]Y[n]\} = E\left\{ (X[n+k]+X[n+k-1])(X[n]+X[n-1]) \right\} = E\{X[n+k]X[n]\} + E\{X[n+k]X[n-1]\} + E\{X[n+k-1]X[n]\} + E\{X[n+k-1]X[n-1]\} \]

so equality. As noted, we have

\[ r_X[k] = r_X[k+1]+r_X[k-1]+r_X[k] \]

\[ = \frac{1}{3}(\delta[k+1]+2\delta[k]+\delta[k-1]). \]

b. The PSD \( R_Y[k] \) is as usual given as the Fourier transform of the ACF:

\[ R_Y[\theta] = \mathcal{F}\{r_Y[k]\} = \frac{1}{3}(e^{i2\pi\theta} + 2 + e^{-i2\pi\theta}) = \frac{2}{3}(1 + \cos(2\pi\theta)) \]

c. To determine the probability \( \Pr\{Y[n]>3/2\} \), we need the PDF \( f_Y[y](y) \). First, we observe that \( Y[n] \) is the sum of two independent samples, \( X[n] \) and \( X[n-1] \). We have

\[ f_Y[y](y) = \int_{-\infty}^{\infty} f_X[x] f_X[x-1](y-x) \, dx = \int_{-\infty}^{\infty} f_X[x](y-x) \, dx = (f_X[x] \ast f_X[x-1])(y), \]

where we have used the independence in the second equality. As noted, we have

\[ f_X[x](x) = f_X[x-1](x) = \frac{1}{2} \text{rect}\left( \frac{x}{2} \right). \]

Thus, we have

\[ f_Y[y](y) = \frac{1}{2} \text{triangle}\left( \frac{y}{2} \right). \]

Graphically:

![Graphical representation](image)

The probability is the integral

\[ \Pr\{Y[n]>3/2\} = \int_{3/2}^{\infty} f_Y[y](y) \, dy = \frac{2}{3} \int_{3/2}^{\infty} \left( 1 - \frac{y}{2} \right) \, dy = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{32}. \]

3 We have a stationary Gaussian process \( X(t) \) with mean zero and ACF \( r_X(\tau) \). We wish to show that the ACF of the process

\[ Y(t) = X^3(t) \]

is

\[ r_Y(\tau) = 9r_X^3(0)r_X(\tau) + 6r_X^2(\tau). \]

We use Prices theorem, T & F page 16, with \( X_1 = X(t), X_2 = X(t+\tau), f(x) = g(x) = x^3 \). For \( n = 1 \) we have

\[ \frac{\partial}{\partial r_X(\tau)} r_Y(\tau) = E\left\{ \frac{\partial^2 X^3(t)}{\partial X(t)} \cdot \frac{\partial X^3(t+\tau)}{\partial X(t+\tau)} \right\} = E\left\{ 3X^2(t) \cdot 3X^2(t+\tau) \right\}. \]

Consider a \( \tau_0 \) such that \( r_X(\tau_0) = 0 \) holds. Then \( X(t) \) and \( X(t+\tau_0) \) are uncorrelated. But, \( (X(t), X(t+\tau_0)) \) is a twodimensional jointly Gaussian variable since the process is a Gaussian process. Uncorrelated variables constituting a two-dimensional jointly Gaussian variable are independent. So, \( X(t) \) and \( X(t+\tau_0) \) are independent. For this particular \( \tau_0 \), we therefore have

\[ \frac{\partial}{\partial r_X(\tau_0)} r_Y(\tau_0) = 9E\left\{ 3X^2(t) \right\} \cdot E\left\{ X^2(t+\tau_0) \right\} = 9r_X^3(0). \]

(1)

For \( n = 2 \) we have

\[ \frac{\partial^2}{\partial r_X(\tau)^2} r_Y(\tau) = E\left\{ \frac{\partial^2 X^3(t)}{\partial X^2(t)} \cdot \frac{\partial^2 X^3(t+\tau)}{\partial X^2(t+\tau)} \right\} = E\left\{ 6X(t) \cdot 6X(t+\tau_0) \right\} = 36r_X^2(\tau). \]

Integrate both sides with respect to \( r_X(\tau) \), and we get

\[ \frac{\partial}{\partial r_X(\tau)} r_Y(\tau) = 18r_X^2(\tau) + C_1, \]

(2)

for some constant \( C_1 \). Again, consider a \( \tau_0 \) such that \( r_X(\tau_0) = 0 \) holds. Of course, \( X(t) \) and \( X(t+\tau_0) \) are still independent. The Equations 1 and 2 then give us

\[ C_1 = 9r_X^3(0). \]

We identify in Equation 2, and integrate both sides with respect to \( r_X(\tau) \) again, and we get

\[ r_Y(\tau) = 6r_X^3(\tau) + 9r_X^2(0)r_X(\tau) + C_2, \]

for some constant \( C_2 \). For a third time, we consider a \( \tau_0 \) such that \( r_X(\tau_0) = 0 \) holds. Again, \( X(t) \) and \( X(t+\tau_0) \) are independent, and we get for this particular \( \tau_0 \),

\[ C_2 = r_Y(\tau_0) = E\left\{ X^3(t) \right\} \cdot E\left\{ X^3(t+\tau_0) \right\} = 0, \]
where we have used the fact that an odd moment of a Gaussian variable with mean zero is zero. Finally, this gives us
\[ r_Y(\tau) = 6r_X^3(\tau) + 9r_X^2(0)r_X(\tau), \]
which is what we wanted to prove.

4

\( Z(t) \) is not WSS, which we can see by calculating \( r_Z(0, 1/f_0) \) and \( r_Z(-3/(4f_0), 1/(4f_0)) \). We have
\[
\begin{align*}
r_Z(0, 1/f_0) &= E\{Z(0)Z(1/f_0)\} \\
&= E\{X(0)X(1/f_0)\} \cos(\pi) \\
&= E\{X(0)X(1/f_0)\} = r_X(1/f_0) \neq 0
\end{align*}
\]
and
\[
\begin{align*}
r_Z(-3/(4f_0), 1/(4f_0)) &= E\{Z(-3/(4f_0))Z(1/(4f_0))\} \\
&= E\{X(-3/(4f_0))X(1/(4f_0))\} \cos(-3\pi/2) \cos(\pi/2) \\
&= 0.
\end{align*}
\]
This gives us \( r_Z(0, 1/f_0) \neq r_Z(-3/(4f_0), 1/(4f_0)) \) which means that \( Z(t) \) is not WSS since
\[ 1/f_0 - 0 = (1/(4f_0) - (-3/(4f_0)). \]
The signal \( Y(t) \) has ACF
\[
\begin{align*}
r_Y(t, t + \tau) &= E\{Y(t)Y(t + \tau)\} \\
&= E\{X(t)X(t + \tau)\} \times \\
&\quad \times \cos(2\pi(f_0t + \Psi)) \cos(2\pi(f_0(t + \tau) + \Psi)) \\
&= r_X(\tau)E\{\cos(2\pi f_0(t + \tau) + \Psi)) \cos(2\pi(f_0(t + \tau) + \Psi))\} \\
&= r_X(\tau)E\{\frac{1}{2}\cos(2\pi f_0(t + 2\Psi)) + \cos(2\pi f_0\tau))\} \\
&= r_X(\tau)\frac{1}{2}\left( \int_0^1 \cos(2\pi f_0(t + 2\psi))d\psi + \right. \\
&\quad + \left. \int_0^1 \cos(2\pi f_0\tau)d\psi \right) \\
&= \frac{1}{2}r_X(\tau) \cos(2\pi f_0\tau).
\end{align*}
\]
This shows us that the ACF is independent of time-shifts. The average of \( Y(t) \) is
\[
\begin{align*}
m_Y(t) &= E\{Y(t)\} \\
&= E\{X(t) \cos(2\pi(f_0t + \Psi))\} \\
&= m_X E\{\cos(2\pi(f_0t + \Psi))\} \\
&= m_X(\tau) \int_0^1 \cos(2\pi f_0(t + \psi)d\psi = 0
\end{align*}
\]
which also is independent of time. This shows us that \( Y(t) \) is stationary in the weak sense with mean \( m_Y = 0 \) and ACF \( r_Y = \frac{1}{2}r_X(\tau) \cos(2\pi f_0\tau). \)

5

We are given the process
\[ X(t) = A + B(t), \]
where \( A \) is a Gaussian variable with non-zero variance \( \sigma_A^2 \), and where \( B(t) \) is a WSS Gaussian process with mean zero and ACF \( r_B(\tau) = R_0 \sin(2\Omega \tau). \) Finally, \( A \) and \( B(t) \) are independent for all time instances \( t. \)

a. We want to determine if \( B(t) \) is SSS.
\( B(t) \) is WSS and Gaussian. A WSS Gaussian process is SSS. So, yes! \( B(t) \) is SSS.

b. We want to determine if \( B(t) \) is ergodic.
The given ACF \( r_B(\tau) \) is a scaled sinc. Then the PSD \( R_B(f) \) is a scaled rect. Specifically, there are no impulses in \( R_B(f). \) Moreover, the Gaussian process \( B(t) \) is stationary with mean zero. Then \( B(t) \) is ergodic.

c. We want to determine if \( X(t) \) is WSS.
First the mean, given by
\[ m_X(t) = E\{A + B(t)\} = m_A + m_B = m_A, \]
which is obviously constant since \( A \) is a stochastic variable. Here we have used that \( m_B = 0. \)
Then the ACF, given by
\[
\begin{align*}
r_X(t, t + \tau, t) &= E\\{\cos(2\pi f_0(t + \tau) + \Psi)) \cos(2\pi(f_0(t + \tau) + \Psi))\} \\
&= E\{A^2\} + 2m_A m_B + r_B(\tau) \\
&= E\{A^2\} + r_B(\tau),
\end{align*}
\]
which does not depend on \( t. \) Again we have used that \( m_B \) is zero.
Yes, \( X(t) \) is WSS.

d. We want to determine if \( X(t) \) is SSS.
Yes, \( X(t) \) is Gaussian and WSS. Thus, it is SSS.
e. We want to determine if \( X(t) \) is ergodic.
No, the time-average of \( X(t) \) will be close to the realization of \( A, \) which with probability one is not \( m_X \) since \( \sigma_A^2 \) is non-zero.
The input can be written as
\[ x[n_1, n_2] = \delta[n_1 - 1, n_2] + \delta[n_1, n_2 - 1] \]
and the impulse response of the system can be written as
\[ h[n_1, n_2] = \delta[n_1 + 1, n_2 - 1] + \delta[n_1, n_2 - 1] + \delta[n_1 - 1, n_2 - 1] + \delta[n_1 - 1, n_2 - 1], \]
where \( \delta[n_1, n_2] \) is the two-dimensional space-discrete unit impulse.

\[ a. \] The output \( y[n_1, n_2] \) is given by the convolution
\[ y[n_1, n_2] = (x \ast h)[n_1, n_2]. \]
We note that \( \delta[n_1, n_2] = \delta[n_1] \delta[n_2] \)
holds, where \( \delta[n] \) is the one-dimensional time-discrete unit impulse. A single two-dimensional impulse is thus separable, and if we convolve two shifted two-dimensional impulses, we get the product of two convolutions of shifted one-dimensional impulses. If we convolve two shifted one-dimensional impulses, the result is a shifted impulse, where the shift is the sum of the two shifts. Therefore, the result of a convolution of two shifted two-dimensional impulses is a shifted two-dimensional impulse, where the shift is the sum of the shifts of the two convolved impulses. Finally, this means that the output of our system is
\[
y[n_1, n_2] = (x \ast h)[n_1, n_2] \\
= \delta[n_1 + 1, n_2 - 2] + \delta[n_1, n_2 - 2] + \delta[n_1 - 1, n_2 - 2] \\
+ \delta[n_1, n_2 - 1] + \delta[n_1 - 1, n_2 - 1] + \delta[n_1 - 2, n_2 - 1] \\
= \begin{cases} 
1, & (n_1, n_2) \in \{(-1, 2), (0, 2), (1, 2), (0, 1), (1, 1), (2, 1)\} \\
0, & \text{elsewhere}
\end{cases}
\]

\[ b. \] We Fourier transform \( x[n_1, n_2] \) and \( h[n_1, n_2] \) and get
\[
X[\theta_1, \theta_2] = \sum_{n_1} \sum_{n_2} x[n_1, n_2] e^{-j2\pi(\theta_1 n_1 + \theta_2 n_2)} \\
= e^{j2\pi(-\theta_1)} + e^{j2\pi(-\theta_2)},
\]
\[
H[\theta_1, \theta_2] = \sum_{n_1} \sum_{n_2} h[n_1, n_2] e^{-j2\pi(\theta_1 n_1 + \theta_2 n_2)} \\
= e^{j2\pi(\theta_1 - \theta_2)} + e^{j2\pi(-\theta_2)} + e^{j2\pi(-\theta_1 - \theta_2)}.
\]
The spectrum of the output is then given by
\[
Y[\theta_1, \theta_2] = X[\theta_1, \theta_2] H[\theta_1, \theta_2] \\
= \left(e^{j2\pi(-\theta_1)} + e^{j2\pi(-\theta_2)}\right) \\
\cdot \left(e^{j2\pi(\theta_1 - \theta_2)} + e^{j2\pi(-\theta_2)} + e^{j2\pi(-\theta_1 - \theta_2)}\right) \\
= e^{j2\pi(-\theta_2)} + e^{j2\pi(-\theta_1 - \theta_2)} + e^{j2\pi(-2\theta_1 - \theta_2)} \\
+ e^{j2\pi(\theta_1 - 2\theta_2)} + e^{j2\pi(-2\theta_2)} + e^{j2\pi(-\theta_1 - 2\theta_2)}
\]

After inverse transformation, we get
\[
y[n_1, n_2] = \begin{cases} 
1, & (n_1, n_2) \in \{(-1, 2), (0, 2), (1, 2), (0, 1), (1, 1), (2, 1)\} \\
0, & \text{elsewhere}
\end{cases}
\]