TSDT14 Signal Theory

Solutions to the exam 2019-01-08

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1

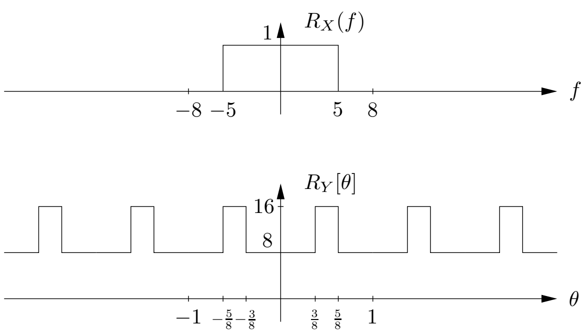
These subtasks have to be treated correctly as partial fulfillment to pass the exam.

a. A stochastic process is said to be *exactly predictable* if it is enough to observe a realization during a finite interval to determine the complete realization.

b. The PSD $R_Y[\theta]$ is given by Poisson's summation formula for stochastic processes (T & F p. 11):

$$R_Y[\theta] = f_s \sum_m R_X(f_s(\theta - m))$$

Graphically:



c. According to T & F, page 15, we have

$$r_Y[k] = 6r_X^3[k] + 9r_X^2[0]r_X[k]$$

For the given ACF, we have $r_X[0] = 1$. Thus, we have

$$r_Y[k] = 6\left(\frac{1}{3}\right)^{3|k|} + 9\left(\frac{1}{3}\right)^{|k|},$$

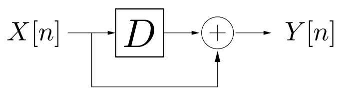
which can be rewritten as

$$r_Y[k] = 6\left(\frac{1}{27}\right)^{|k|} + 9\left(\frac{1}{3}\right)^{|k|}$$

if you are amused by that.

2

We are given the following system, where $X[n]$ is a sequence of independent random variables, uniformly distributed on the interval $[-1, 1]$.



For the input, we have the PDF

$$f_{X[n]}(x) = \frac{1}{2}\,\mathrm{rect}\left(\frac{x}{2}\right).$$

and the mean

$$m_X = \int_{-\infty}^{\infty} x f_{X[n]}(x)\,dx = \int_{-1}^{1} \frac{x}{2}\,dx = 0.$$

To determine the ACF for the input, we study the cases $k = 0$ and $k \neq 0$ separately. For $k = 0$ we get

$$r_X[0] = \mathrm{E}\{X^2[n]\} = \int_{-\infty}^{\infty} x^2 f_{X[n]}(x)\,dx = \int_{-1}^{1} \frac{x^2}{2}\,dx = \frac{1}{3},$$

and for $k \neq 0$ we have

$$\begin{aligned} r_X[k] &= \mathrm{E}\{X[n+k]X[n]\} \\ &= \mathrm{E}\{X[n+k]\}\mathrm{E}\{X[n]\} = m_X^2 = 0, \end{aligned}$$

where we have used the independence in the second equality. Totally, we have

$$r_X[k] = \frac{1}{3}\delta[k].$$

a. To determine the ACF $r_Y[k]$, we notice from the figure that we have

$$Y[n] = X[n] + X[n-1].$$

The ACF is then given by

$$\begin{aligned} r_Y[k] &= \mathrm{E}\{Y[n+k]Y[n]\} = \\ &= \mathrm{E}\Big\{\big(X[n+k]+X[n+k-1]\big)\big(X[n]+X[n-1]\big)\Big\} \\ &= \mathrm{E}\{X[n+k]X[n]\} + \mathrm{E}\{X[n+k]X[n-1]\} + \\ &\quad + \mathrm{E}\{X[n+k-1]X[n]\} + \mathrm{E}\{X[n+k-1]X[n-1]\} \\ &= r_X[k] + r_X[k+1] + r_X[k-1] + r_X[k] \\ &= \frac{1}{3}\big(\delta[k+1] + 2\delta[k] + \delta[k-1]\big). \end{aligned}$$

b. The PSD $R_Y[k]$ is as usual given as the Fourier transform of the ACF:

$$\begin{aligned} R_Y[\theta] &= \mathcal{F}\{r_Y[k]\} = \\ &= \frac{1}{3}\big(e^{j2\pi\theta} + 2 + e^{-j2\pi\theta}\big) = \frac{2}{3}\big(1 + \cos(2\pi\theta)\big). \end{aligned}$$

c. To determine the probability $\Pr\{Y[n] > 3/2\}$, we need the PDF $f_{Y[n]}(y)$. First, we observe that $Y[n]$ is the sum of two independent samples, $X[n]$ and $X[n-1]$. We have

$$f_{Y[n]}(y) = \int_{-\infty}^{\infty} f_{X[n],X[n-1]}(x, y-x)\,dx =$$

$$= \int_{-\infty}^{\infty} f_{X[n]}(x) f_{X[n-1]}(y-x)\,dx = (f_{X[n]} * f_{X[n-1]})(y),$$

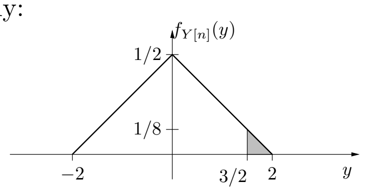
where we have used the independence in the second equality. As noted, we have

$$f_{X[n]}(x) = f_{X[n-1]}(x) = \frac{1}{2}\,\mathrm{rect}\left(\frac{x}{2}\right).$$

Thus, we have

$$f_{Y[n]}(y) = \frac{1}{2}\,\mathrm{triangle}\left(\frac{y}{2}\right).$$

Graphically:



The probability is the integral

$$\Pr\{Y[n] > 3/2\} = \int_{3/2}^{\infty} f_{Y[n]}(y)\,dy =$$

$$= \int_{3/2}^{2} \frac{1}{2}\left(1 - \frac{y}{2}\right)\,dy = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{8} = \frac{1}{32}.$$

3

We have a stationary Gaussian process $X(t)$ with mean zero and ACF $r_X(\tau)$. We wish to show that the ACF of the process

$$Y(t) = X^3(t)$$

is

$$r_Y(\tau) = 9r_X^2(0)r_X(\tau) + 6r_X^3(\tau).$$

We use Prices theorem, T & F page 16, with $X_1 = X(t)$, $X_2 = X(t+\tau)$, $f(x) = g(x) = x^3$. For $n = 1$ we have

$$\begin{aligned} \frac{\partial}{\partial r_X(\tau)} r_Y(\tau) &= \mathrm{E}\left\{\frac{\partial X^3(t)}{\partial X(t)} \cdot \frac{\partial X^3(t+\tau)}{\partial X(t+\tau)}\right\} \\ &= \mathrm{E}\Big\{3X^2(t)\cdot 3X^2(t+\tau)\Big\}. \end{aligned}$$

Consider a τ_0 such that $r_X(\tau_0) = 0$ holds. Then $X(t)$ and $X(t+\tau_0)$ are uncorrelated. But, $(X(t), X(t+\tau_0))$ is a twodimensional jointly Gaussian variable since the process is a Gaussian process. Uncorrelated variables constituting a two-dimensional jointly Gaussian variable are independent. So, $X(t)$ and $X(t+\tau_0)$ are independent. For this particular τ_0, we therefore have

$$\frac{\partial}{\partial r_X(\tau_0)} r_Y(\tau_0) = 9\mathrm{E}\Big\{3X^2(t)\Big\}\cdot\mathrm{E}\Big\{X^2(t+\tau_0)\Big\} = 9r_X^2(0). \quad (1)$$

For $n = 2$ we have

$$\begin{aligned} \frac{\partial^2}{\partial r_X^2(\tau)} r_Y(\tau) &= \mathrm{E}\left\{\frac{\partial^2 X^3(t)}{\partial X^2(t)} \cdot \frac{\partial^2 X^3(t+\tau)}{\partial X^2(t+\tau)}\right\} \\ &= \mathrm{E}\{6X(t)\cdot 6X(t+\tau)\} = 36 r_X(\tau). \end{aligned}$$

Integrate both sides with respect to $r_X(\tau)$, and we get

$$\frac{\partial}{\partial r_X(\tau)} r_Y(\tau) = 18r_X^2(\tau) + C_1, \quad (2)$$

for some constant C_1. Again, we consider a τ_0 such that $r_X(\tau_0) = 0$ holds. Of course, $X(t)$ and $X(t+\tau_0)$ are still independent. The Equations 1 and 2 then give us

$$C_1 = 9r_X^2(0).$$

We identify in Equation 2, and integrate both sides with respect to $r_X(\tau)$ again, and we get

$$r_Y(\tau) = 6r_X^3(\tau) + 9r_X^2(0)r_X(\tau) + C_2,$$

for some constant C_2. For a third time, we consider a τ_0 such that $r_X(\tau_0) = 0$ holds. Again, $X(t)$ and $X(t+\tau_0)$ are independent, and we get for this particular τ_0,

$$C_2 = r_Y(\tau_0) = \mathrm{E}\Big\{X^3(t)\Big\}\,\mathrm{E}\Big\{X^3(t+\tau_0)\Big\} = 0,$$

where we have used the fact that an odd moment of a Gaussian variable with mean zero is zero. Finally, this gives us

$$r_Y(\tau) = 6r_X^3(\tau) + 9r_X^2(0)r_X(\tau),$$

which is what we wanted to prove.

4

$Z(t)$ is not WSS, which we can see by calculating $r_Z(0, 1/f_0)$ and $r_Z(-3/(4f_0), 1/(4f_0))$. We have

$$\begin{aligned} r_Z(0, 1/f_0) &= E\{Z(0)Z(1/f_0)\} \\ &= E\{X(0)X(1/f_0)\cos 0 \cos(\pi)\} \\ &= E\{X(0)X(1/f_0)\} = r_X(1/f_0) \neq 0 \end{aligned}$$

and

$$\begin{aligned} & r_Z(-3/(4f_0), 1/(4f_0)) = \\ &= E\{Z(-3/(4f_0))Z(1/(4f_0))\} \\ &= E\{X(-3/(4f_0))X(1/(4f_0))\cos(-3\pi/2)\cos(\pi/2)\} \\ &= 0. \end{aligned}$$

This gives us $r_Z(0, 1/f_0) \neq r_Z(-3/(4f_0), 1/(4f_0))$ which means that $Z(t)$ is not WSS since

$$1/f_0 - 0 = 1/(4f_0) - (-3/(4f_0)).$$

The signal $Y(t)$ has ACF

$$\begin{aligned} & r_Y(t, t+\tau) = \\ &= E\{Y(t)Y(t+\tau)\} \\ &= E\{X(t)X(t+\tau))\times \\ &\quad \times \cos(2\pi(f_0 t + \Psi))\cos(2\pi(f_0(t+\tau) + \Psi))\} \\ &= r_X(\tau)E\{\cos(2\pi(f_0 t + \Psi))\cos(2\pi(f_0(t+\tau) + \Psi))\} \\ &= r_X(\tau)E\{\frac{1}{2}(\cos(2\pi(f_0(2t+\tau) + 2\Psi)) + \cos(2\pi f_0\tau))\} \\ &= r_X(\tau)\frac{1}{2}\bigg(\int_0^1 \cos(2\pi(f_0(2t+\tau) + 2\psi))d\psi + \\ &\quad + \int_0^1 \cos(2\pi f_0\tau)d\psi\bigg) \\ &= \frac{1}{2}r_X(\tau)\cos(2\pi f_0\tau) \end{aligned}$$

This shows us that the ACF is independent of time-shifts. The average of $Y(t)$ is

$$\begin{aligned} m_Y(t) &= E\{Y(t)\} \\ &= E\{X(t)\cos(2\pi(f_0 t + \Psi))\} \\ &= m_X E\{\cos(2\pi(f_0 t + \Psi))\} \\ &= m_X(\tau)\int_0^1 \cos(2\pi(f_0 t + \psi))d\psi = 0 \end{aligned}$$

which also is independent of time. This shows us that $Y(t)$ is stationary in the weak sense with mean $m_Y = 0$ and ACF $r_Y = \frac{1}{2}r_X(\tau)\cos(2\pi f_0\tau)$.

5

We are given the process

$$X(t) = A + B(t),$$

where A is a Gaussian variable with non-zero variance σ_A^2, and where $B(t)$ is a WSS Gaussian process with mean zero and ACF $r_B(\tau) = R_0 \operatorname{sinc}(20\tau)$. Finally, A and $B(t)$ are independent for all time instances t.

a. We want to determine if $B(t)$ is SSS.

$B(t)$ is WSS and Gaussian. A WSS Gaussian process is SSS. So, yes! $B(t)$ is SSS.

b. We want to determine if $B(t)$ is ergodic.

The given ACF $r_B(\tau)$ is a scaled sinc. Then the PSD $R_B(f)$ is a scaled rect. Specifically, there are no impulses in $R_B(f)$. Moreover, the Gaussian process $B(t)$ is stationary with mean zero. Then $B(t)$ is ergodic.

c. We want to determine if $X(t)$ is WSS.

First the mean, given by

$$m_X(t) = \mathrm{E}\{A + B(t)\} = m_A + m_B = m_A,$$

which is obviously constant since A is a stochastic variable. Here we have used that m_B is zero.

Then the ACF, given by

$$\begin{aligned} r_X(t+\tau, t) &= \mathrm{E}\Big\{\big(A + B(t+\tau)\big)\big(A + B(t)\big)\Big\} \\ &= \mathrm{E}\{A^2\} + 2m_A m_B + r_B(\tau) \\ &= \mathrm{E}\{A^2\} + r_B(\tau), \end{aligned}$$

which does not depend on t. Again we have used that m_B is zero.

Yes, $X(t)$ is WSS.

d. We want to determine if $X(t)$ is SSS.

Yes, $X(t)$ is Gaussian and WSS. Thus, it is SSS.

e. We want to determine if $X(t)$ is ergodic.

No, the time-average of $X(t)$ will be close to the realization of A, which with probability one is not m_X since σ_A^2 is non-zero.

6

The input can be written as

$$x[n_1,n_2] = \delta[n_1-1,n_2] + \delta[n_1,n_2-1]$$

and the impulse response of the system can be written as

$$h[n_1,n_2] = \delta[n_1+1,n_2-1]+\delta[n_1,n_2-1]+\delta[n_1-1,n_2-1],$$

where $\delta[n_1,n_2]$ is the two-dimensional space-discrete unit impulse.

a. The output $y[n_1,n_2]$ is given by the convolution

$$y[n_1,n_2] = (x \circledast h)[n_1,n_2].$$

We note that

$$\delta[n_1,n_2] = \delta[n_1]\delta[n_2]$$

holds, where $\delta[n]$ is the one-dimensional time-discrete unit impulse. A single two-dimensional impulse is thus separable, and if we convolve two shifted two-dimensional impulses, we get the product of two convolutions of shifted one-dimensional impulses. If we convolve two shifted one-dimensional impulses, the result is a shifted impulse, where the shift is the sum of the two shifts. Therefore, the result of a convolution of two shifted two-dimensional impulses is a shifted two-dimensional impulse, where the shift is the sum of the shifts of the two convolved impulses. Finally, this means that the output of our system is

$$\begin{aligned} y[n_1,n_2] &= (x \circledast h)[n_1,n_2] \\ &= \delta[n_1+1,n_2-2] + \delta[n_1,n_2-2] + \delta[n_1-1,n_2-2] \\ &+ \delta[n_1,n_2-1] + \delta[n_1-1,n_2-1] + \delta[n_1-2,n_2-1] \\ &= \begin{cases} 1, & (n_1,n_2)\in\{(-1,2),(0,2),(1,2),(0,1),(1,1),(2,1)\} \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

b. We Fourier transform $x[n_1,n_2]$ and $h[n_1,n_2]$ and get

$$\begin{aligned} X[\theta_1,\theta_2] &= \sum_{n_1}\sum_{n_2} x[n_1,n_2] e^{-j2\pi(\theta_1 n_1+\theta_2 n_2)} \\ &= e^{j2\pi\cdot(-\theta_1)} + e^{j2\pi\cdot(-\theta_2)}, \\ H[\theta_1,\theta_2] &= \sum_{n_1}\sum_{n_2} h[n_1,n_2] e^{-j2\pi(\theta_1 n_1+\theta_2 n_2)} \\ &= e^{j2\pi\cdot(\theta_1-\theta_2)} + e^{j2\pi\cdot(-\theta_2)} + e^{j2\pi\cdot(-\theta_1-\theta_2)}. \end{aligned}$$

The spectrum of the output is then given by

$$\begin{aligned} Y[\theta_1,\theta_2] &= X[\theta_1,\theta_2]H[\theta_1,\theta_2] \\ &= \left(e^{j2\pi\cdot(-\theta_1)} + e^{j2\pi\cdot(-\theta_2)}\right) \\ &\cdot \left(e^{j2\pi\cdot(\theta_1-\theta_2)} + e^{j2\pi\cdot(-\theta_2)} + e^{j2\pi\cdot(-\theta_1-\theta_2)}\right) \\ &= e^{j2\pi\cdot(-\theta_2)} + e^{j2\pi\cdot(-\theta_1-\theta_2)} + e^{j2\pi\cdot(-2\theta_1-\theta_2)} \\ &+ e^{j2\pi\cdot(\theta_1-2\theta_2)} + e^{j2\pi\cdot(-2\theta_2)} + e^{j2\pi\cdot(-\theta_1-2\theta_2)} \end{aligned}$$

After inverse transformation, we get

$$y[n_1,n_2] = \begin{cases} 1, & (n_1,n_2)\in\{(-1,2),(0,2),(1,2),(0,1),(1,1),(2,1)\} \\ 0, & \text{elsewhere} \end{cases}$$

just as before.