TSDT14 Signal Theory

Solutions to the exam 2018-01-03

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At least two of the following three sub-tasks have to be treated correctly as partial fulfillment to pass the exam:

a. The following two answers have been accepted:

i) Let $\overline{t}$ and $\overline{u}$ be given by

$$\begin{aligned}\overline{t} &\triangleq (t_1, t_2, \ldots, t_N),\\ \overline{u} &\triangleq (t_1+\Delta t,\ t_2+\Delta t,\ \ldots,\ t_N+\Delta t).\end{aligned}$$

The time-continuous stochastic process $X(t)$ is said to be *stationary in the strict sense* if

$$F_{X(\overline{t})}(\overline{x}) = F_{X(\overline{u})}(\overline{x})$$

holds for any positive integer N, any choice of $\overline{t}$ and any Δt.

ii) A time-continuous process is said to be *stationary in the strict sense* if its statistical properties are invariant to a time shift.

b. The PSD of the output is given by

$$\begin{aligned}R_Y[\theta] &= |H[\theta]|^2 R_X[\theta]\\ &= \begin{cases} 4\big(1+\cos(2\pi\theta)\big), & |\theta| < 0.1,\\ 0, & 0.1 \le |\theta| \le 0.5,\end{cases}\end{aligned}$$

periodically repeated with period 1.

c. According to the Tables and Formulas p 15, we have for this case

$$r_Y[k] = r_X[0] + 2r_X^2[k] = 1 + \frac{2}{(1+k^2)^2}$$

2

We are given $X(t)$, which is a time-continuous Gaussian process with mean $m_X = 1$ and ACF $r_X(\tau) = 1 + \operatorname{sinc}(\tau)$. This process is the input to an LTI system with frequency response $H(f) = 2\cdot\operatorname{rect}(2f)$, and $Y(t)$ is the output.

a. According to T&F, page 9, we have

$$m_Y = H(0)m_X = 2.$$

c. From T&F, pages 18 and 19, we have

$$R_X(f) = \delta(f) + \operatorname{rect}(f).$$

According to T&F, page 9, we have

$$\begin{aligned}R_Y(f) &= \big|H(f)|^2 R_X(f)\\ &= 4\operatorname{rect}(2f)\big(\delta(f) + \operatorname{rect}(f)\big)\\ &= 4\delta(f) + 4\operatorname{rect}(2f).\end{aligned}$$

b. According to T&F, page 7, we have

$$r_Y(\tau) = \mathcal{F}^{-1}\big\{R_Y(f)\big\} = 4 + 2\operatorname{sinc}(\tau/2).$$

d. According to T&F, page 7, we have

$$P_Y = r_Y(0) = 6.$$

e. According to T&F, page 2, we have

$$\sigma_Y^2 = P_Y - m_Y^2 = 2.$$

The standard trick to transform any Gaussian variable into a Gaussian variable with mean 0 and variance 1, gives us

$$\Pr\big\{Y(t) > 2\big\} = Q\left(\frac{2-m_Y}{\sigma_Y}\right) = Q(0) = 1/2.$$

3

We have the time-discrete process $X[n]$ with PSD

$$\begin{aligned}R_X[\theta] &= 2 + \cos(4\pi\theta) + \cos(8\pi\theta)\\ &= 2 + \frac{e^{j4\pi\theta} + e^{-j4\pi\theta}}{2} + \frac{e^{j8\pi\theta} + e^{-j8\pi\theta}}{2}.\end{aligned}$$

The corresponding ACF is

$$\begin{aligned}r_X[k] = 2 &+ \frac{1}{2}\delta[k+4] + \frac{1}{2}\delta[k+2] +\\ &+ 2\delta[k] + \frac{1}{2}\delta[k-2] + \frac{1}{2}\delta[k-4].\end{aligned}$$

Specifically, we note that we have

$$r_X[0] = 2, \qquad r_X[1] = 0, \qquad r_X[2] = 1/2.$$

We are supposed to use the prediction

$$\hat{X}[n_0+1] = aX[n_0] + bX[n_0-1].$$

We want to minimize

$$\begin{aligned}
\epsilon^2 &= \mathrm{E}\Big\{\big(\hat{X}[n_0{+}1] - X[n_0{+}1]\big)^2\Big\}\\
&= \mathrm{E}\Big\{\big(aX[n_0] + bX[n_0{-}1] - X[n_0{+}1]\big)^2\Big\}\\
&= a^2\mathrm{E}\Big\{X^2[n_0]\Big\} + b^2\mathrm{E}\Big\{X^2[n_0{-}1]\Big\} + \mathrm{E}\Big\{X^2[n_0{+}1]\Big\}\\
&\quad + 2ab\mathrm{E}\Big\{X[n_0]X[n_0{-}1]\Big\} - 2a\mathrm{E}\Big\{X[n_0]X[n_0{+}1]\Big\}\\
&\quad - 2b\mathrm{E}\Big\{X[n_0{-}1]X[n_0{+}1]\Big\}\\
&= (a^2+b^2+1)r_X[0] + 2a(b-1)r_X[1] - 2br_X[2]\\
&= 2(a^2+b^2+1) - b.
\end{aligned}$$

The above function obviously has only one extreme point, and that is a minimum. It is found by partial differentiation as

$$\frac{\partial\epsilon^2}{\partial a} = 4a, \qquad \frac{\partial\epsilon^2}{\partial b} = 4b-1,$$

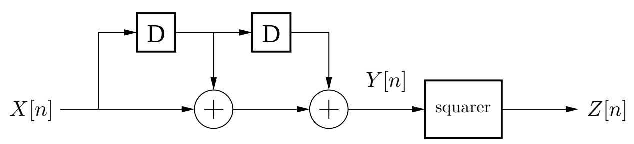
Setting those to zero gives us the solution

$$a = 0, \qquad b = 1/4,$$

which is our answer.

4

We have the system below, where the input $X[n]$ is a sequence of independent stochastic variables.



Our task is to determine the ACF $r_Z[k]$ and the PSD $R_Z[\theta]$.

a. The input $X[n]$ is Gaussian with mean $m_X = 0$ and variance $\sigma_X^2 = 1$. Thus, we have

$$r_X[0] = \mathrm{E}\{X^2[n]\} = \sigma_X^2 + m_X^2 = 1,$$

and for $k \neq 0$ we have

$$\begin{aligned}
r_X[k] &= \mathrm{E}\{X[n]X[n+k]\}\\
&= \mathrm{E}\{X[n]\}\mathrm{E}\{X[n+k]\}\\
&= m_X^2 = 0,
\end{aligned}$$

and thus totally

$$r_X[k] = \delta[k] \qquad \text{and} \qquad R_X[\theta] = 1.$$

The initial filter has impulse response

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

and frequency response

$$H[\theta] = 1 + e^{-j2\pi\theta} + e^{-j4\pi\theta}.$$

Then its output $Y[n]$ has PSD

$$\begin{aligned}
R_Y[\theta] &= \big|H[\theta]\big|^2 R_X[\theta]\\
&= (1 + e^{-j2\pi\theta} + e^{-j4\pi\theta})(1 + e^{j2\pi\theta} + e^{j4\pi\theta})\\
&= 3 + 4\cos(2\pi\theta) + 2\cos(4\pi\theta)
\end{aligned}$$

and thus ACF

$$\begin{aligned}
r_Y[k] &=\\
&= \delta[k{+}2] + 2\delta[k{+}1] + 3\delta[k] + 2\delta[k{-}1] + \delta[k{-}2].
\end{aligned}$$

T&F, p. 15 ($Y[n]$ Gaussian with $m_Y = 0$) gives us the ACF

$$\begin{aligned}
r_Z[k] &= 2r_Y^2[k] + r_Y^2[0] =\\
&= 2\delta[k+2] + 8\delta[k+1] + 18\delta[k]+\\
&\quad + 8\delta[k-1] + 2\delta[k-2] + 9,
\end{aligned}$$

and consequently the PSD

$$\begin{aligned}
R_Z[\theta] &=\\
&= 2e^{j4\pi\theta} + 8e^{j2\pi\theta} + 18 + 8e^{-j2\pi\theta} + 2e^{-j4\pi\theta}+\\
&\quad + 9\sum_m \delta(\theta-m)\\
&= 4\cos(4\pi\theta) + 16\cos(2\pi\theta) + 18 + 9\sum_m \delta(\theta-m).
\end{aligned}$$

b. The input $X[n]$ is binary, taking values ± 1 with equal probability, which similarily as in **a** gives us

$$r_X[k] = \delta[k] \qquad \text{and} \qquad R_X[\theta] = 1.$$

Here, we need to analyze the situation using definitions. Notice that we have $X^2[n] = 1$. From the figure, we find that we have

$$Y[n] = X[n] + X[n-1] + X[n-2],$$

and

$$\begin{aligned}
Z[n] = Y^2[n] &= \big(X[n] + X[n-1] + X[n-2]\big)^2\\
&= X^2[n] + X^2[n-1] + X^2[n-2]+\\
&\quad + 2\big(X[n]X[n-1] + X[n-1]X[n-2]+\\
&\quad + X[n]X[n-2]\big)\\
&= 3 + 2\big(X[n]X[n-1] + X[n-1]X[n-2]+\\
&\quad + X[n]X[n-2]\big)
\end{aligned}$$

By definition (T&F, p. 7), we have

$$
\begin{aligned}
r_Y[k] &= \mathrm{E}\big\{Z[n+k]Z[n]\big\} = \\
&= \mathrm{E}\Big\{\Big(3+2\big(X[n+k]X[n+k-1]+ \\
&\quad + X[n+k-1]X[n+k-2]+ \\
&\quad + X[n+k]X[n+k-2]\big)\Big)\cdot \\
&\quad \cdot\Big(3+2\big(X[n]X[n-1]+ \\
&\quad + X[n-1]X[n-2]+X[n]X[n-2]\big)\Big)\Big\} \\
&= 6\mathrm{E}\big\{X[n+k]X[n+k-1]+ \\
&\quad + X[n+k-1]X[n+k-2]+ \\
&\quad + X[n+k]X[n+k-2]\big\} \\
&+ 6\mathrm{E}\big\{X[n]X[n-1]+X[n-1]X[n-2]+ \\
&\quad + X[n]X[n-2]\big\} \\
&+ 4\mathrm{E}\Big\{\big(X[n+k]X[n+k-1]+ \\
&\quad + X[n+k-1]X[n+k-2]+ \\
&\quad + X[n+k]X[n+k-2]\big)\cdot \\
&\quad \cdot\big(X[n]X[n-1]+X[n-1]X[n-2]+ \\
&\quad + X[n]X[n-2]\big)\Big\}+9
\end{aligned}
$$

The only terms above that are not zero are those including only squares of samples of the process. Removing all those that are zero for all k, we are left with

$$
\begin{aligned}
&r_Y[k] = \\
&= 4\mathrm{E}\big\{X[n{+}k]X[n{+}k{-}1]X[n]X[n{-}1]\big\} \\
&+ 4\mathrm{E}\big\{X[n{+}k]X[n{+}k{-}1]X[n{-}1]X[n{-}2]\big\} \\
&+ 4\mathrm{E}\big\{X[n{+}k{-}1]X[n{+}k{-}2]X[n]X[n{-}1]\big\} \\
&+ 4\mathrm{E}\big\{X[n{+}k{-}1]X[n{+}k{-}2]X[n{-}1]X[n{-}2]\big\} \\
&+ 4\mathrm{E}\big\{X[n{+}k]X[n{+}k{-}2]X[n]X[n{-}2]\big\}+9 \\
&= 4\big(\delta[k]+\delta[k{+}1]+\delta[k{-}1]+\delta[k]+\delta[k]\big)+9 \\
&= 4\delta[k{+}1]+12\delta[k]+4\delta[k{-}1]+9.
\end{aligned}
$$

The corresponding PSD is

$$
\begin{aligned}
R_Y[\theta] &= 4e^{j2\pi\theta}+12+4e^{-j2\pi\theta}+9\sum_m \delta(\theta-m) \\
&= 8\cos(2\pi\theta)+12+9\sum_m \delta(\theta-m).
\end{aligned}
$$

5

We have the periodic signal

$$
c(t) = \begin{cases} 1, & 0 \le t < T/2, \\ -1, & T/2 \le t < T, \\ c(t+kT), & k \text{ integer}, \end{cases}
$$

with period T.

a. We want to show that $Y(t) = c(t)\cdot X(t)$ is WSS if $X(t)$ is white. Assuming that $X(t)$ is white, we have $r_X(\tau) = R_0\delta(\tau)$ and $m_X = 0$. Then we get the mean

$$
\begin{aligned}
m_Y(t) &= \mathrm{E}\big\{c(t)X(t)\big\} = c(t)\mathrm{E}\big\{X(t)\big\} \\
&= c(t)m_X = 0,
\end{aligned}
$$

which is constant. We also find the ACF

$$
\begin{aligned}
r_Y(t+\tau,t) &= \mathrm{E}\big\{c(t+\tau)X(t+\tau)c(t)X(t)\big\} \\
&= c(t+\tau)c(t)\mathrm{E}\big\{X(t+\tau)X(t)\big\} \\
&= c(t+\tau)c(t)r_X(\tau) \\
&= c(t+\tau)c(t)R_0\delta(\tau) \\
&= c^2(t)R_0\delta(\tau) = R_0\delta(\tau),
\end{aligned}
$$

which is independent of time t. Thus, $Y(t)$ is WSS if $X(t)$ is white. More precisely, $Y(t)$ is white in this case.

b. We want to show that $Y(t) = c(t)\cdot X(t)$ may be non-stationary if $X(t)$ is non-white. From **a**, we have

$$
m_Y(t) = c(t)m_X,
$$

which depends on t if m_X is non-zero.

We notice that the statement in **a** is true only if we assume the definition of white that demands that the mean is zero.

6

We have the space-continuous signal

$$
x(a_1,a_2) = \begin{cases} 1, & 0 \le a_1 < 2 \text{ and } 0 \le a_2 < 2, \\ 0, & \text{elsewhere}, \end{cases}
$$

which is the input to an LSI system with impulse response

$$
h(a_1,a_2) = \delta(a_1,a_2) + \delta(a_1+1,a_2-1).
$$

Let $y(a_1,a_2)$ denote the output. Then we have

$$
\begin{aligned}
y(a_1,a_2) &= x(a_1,a_2) + x(a_1+1,a_2-1) \\
&= \begin{cases} 2, & 0 \le a_1 < 1 \text{ and } 1 \le a_2 < 2, \\ 1, & -1 \le a_1 < 0 \text{ and } 1 \le a_2 < 3, \\ 1, & 0 \le a_1 < 1 \text{ and } 2 \le a_2 < 3, \\ 1, & 1 \le a_1 < 2 \text{ and } 0 \le a_2 < 2, \\ 1, & 0 \le a_1 < 1 \text{ and } 0 \le a_2 < 1, \\ 0, & \text{elsewhere}. \end{cases}
\end{aligned}
$$