Exam in TSDT14 Signal Theory

Exam code:	TEN1
Date:	2018-10-24 **Time:** 14.00–18.00
Place:	TER2, TERE
Teacher:	Mikael Olofsson, tel: 281343
Visiting exam:	15 and 17
Administrator:	Carina Lindström, 013-284423, carina.e.lindstrom@liu.se
Department:	ISY
Allowed aids:	Olofsson: *Tables and Formulas for Signal Theory* A German 10-Mark note of the fourth series (1991-2001). Pocket calculators of all kinds with empty memory.
Number of tasks:	6
Grading:	Task one is not graded in terms of points. At least two of its three sub-tasks have to be treated correctly as partial fulfillment to pass the exam. Tasks 2-6 yield at most 5 points each. Sloppy solutions and solutions that are hard to read are subject to hard judgement, as are unresonable answers. Totally, you can get 25 points at most. For grade three you need 10 points, for grade four 15 points and for grade five 20 points.
Solutions:	Will be published in the course room no later than three days after the exam, under Previous exams.
Result:	You get a message about your result via an automatic email from Ladok. Note that we cannot file your result if you are not registered on the course. That also means that you will not get an automated email about your result if you are not registered on the course.
Exam return:	In the student office of ISY, Building B, between entrances 27-29, ground floor, Corridor D, right next to Café Java, starting about two weeks after the exam.

1 These sub-tasks have to be treated correctly as partial fulfillment to pass the exam:

a. Define the concept *ensemble average.*

b. The time-discrete signal $X[n]$ has PSD

$$R_X[\theta] = 1 + \cos(2\pi\theta).$$

This signal is pulse-amplitude modulated using the pulse $p(t)$, with spectrum

$$P(f) = \begin{cases} \cos(2\pi f/10), & |f| < 5 \text{ Hz}, \\ 0, & \text{elsewhere} \end{cases}$$

The sampling frequency is 20 Hz. Sketch the PSD of the output. Do not forget to grade the axes.

c. We have a stochastic variable that is uniformly distributed on the interval $[-A, A]$. It is quantized uniformly with 256 quantization steps, and the saturation levels are $\pm A$. Determine the quantization distortion which - as usual - is defined as the expected quadratic error.

2 The time-continuous Gaussian process $X(t)$ with mean $m_X = 0$ and auto-correlation function $r_X(\tau) = 10^3 \cdot \text{sinc}(10^3\tau)$ is the input to an LP filter with frequency response

$$H(f) = \begin{cases} 1 - 10^{-3} \cdot |f|, & |f| < 10^3, \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the probability that the output of the filter has absolute value larger than 5. (5 p)

3 Using two WSS signals $X(t)$ and $Y(t)$, both with mean zero, we form the signal

$$Z(t) = X(t)\cos(2\pi f_c t + \Psi) + Y(t)\sin(2\pi f_c t + \Psi),$$

where f_c is a carrier frequency and Ψ is uniformly distributed on $[0, 2\pi)$. The random phase Ψ is independent of $X(t)$ and of $Y(t)$.

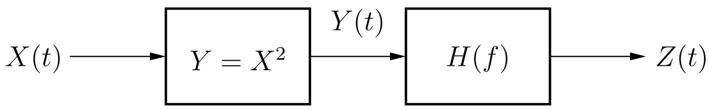
a. Show that $Z(t)$ can be WSS even if $X(t)$ and $Y(t)$ are dependent. What demands have to be fulfilled then? (4 p)

b. Give one explicit example where $X(t)$ and $Y(t)$ are dependent, and where $Z(t)$ is WSS. (Notice that this part can be solved without solving part a) (1 p)

4 The input of the squarer below is band-limited white Gaussian noise with mean zero and power spectral density

$$R_X(f) = \begin{cases} R_0, & f_0 - W \leq |f| \leq f_0 + W, \\ 0, & \text{elsewhere,} \end{cases}$$

where we have $0 < W \ll f_0$.



The frequency response of the filter is

$$H(f) = \begin{cases} 1, & |f| \leq W, \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the power spectral density $R_Z(f)$ of the signal $Z(t)$. Also, determine the power of the output. (5 p)

5 Interleaving is a method that is used in error control coding. In short, it means that samples of a signal are reordered in a structured manner. We are here going to study a very simple type of reordering.

Let $X[n]$ be a WSS time-discrete process. Now consider the new process

$$Y[n] = \begin{cases} X[n+1], & n \text{ even,} \\ X[n-1], & n \text{ odd.} \end{cases}$$

a. Express the ACF of $Y[n]$ in terms of the ACF of $X[n]$. (3 p)

b. Determine under what circumstances $Y[n]$ is WSS. (2 p)

6 Let $X(t)$ be a circular, WSS, complex process. Consider the process $Y(t) = e^{j\phi}X(t)$ for a fixed real-valued ϕ.

a. Show that $X(t)$ and $Y(t)$ have the same PSD. (1 p)

b. Show that $Y(t)$ is circular for all choices of ϕ. (4 p)

Formulas on Complex Processes

Complex process: $X(t)$, where $X_1(t)$ is the real part, and $X_2(t)$ is the imaginary part.

ACF: $r_X(\tau) = \mathrm{E}\{X(t+\tau)X^*(t)\}$

$$= r_{X_1}(\tau) + r_{X_2}(\tau) + j\big(r_{X_1,X_2}(-\tau) - r_{X_1,X_2}(\tau)\big)$$

PSD: $R_X(f) = R_{X_1}(f) + R_{X_2}(f) + j\big(R^*_{X_1,X_2}(f) - R_{X_1,X_2}(f)\big)$

Formulas on Baseband Representation - Deterministic Input

$\tilde{s}(t)$ is the message and $s(t)$ is the modulated signal, with bandwidth of $\tilde{s}(t)$ less than f_c.

Passband: $s(t) = \mathrm{Re}\Big\{\tilde{s}(t)e^{j2\pi f_\mathrm{c} t}\Big\} = s_\mathrm{I}(t)\cos(2\pi f_\mathrm{c} t) - s_\mathrm{Q}(t)\sin(2\pi f_\mathrm{c} t)$

Baseband: $\tilde{s}(t) = s_\mathrm{I}(t) + js_\mathrm{Q}(t)$

Spectrum: $S(f) = \frac{1}{2}\big(\tilde{S}(f-f_\mathrm{c}) + \tilde{S}^*(-f-f_\mathrm{c})\big)$, $\quad \tilde{S}(f) = 2S(f+f_\mathrm{c})u(f+f_\mathrm{c})$

Filtering: $y(t) = (x * h)(t) \quad \Leftrightarrow \quad \tilde{y}(t) = \frac{1}{2}(\tilde{x} * \tilde{h})(t)$
where $\tilde{x}(t)$, $\tilde{h}(t)$ and $\tilde{y}(t)$ are the complex baseband representations of $x(t)$, $h(t)$ and $y(t)$, respectively.

Formulas on Baseband Representation - Stochastic Input

$\tilde{S}(t)$ is the message with $m_{\tilde{S}} = 0$ and $S(t)$ is the modulated signal. The bandwidth of $\tilde{S}(t)$ must be less than f_c.

Passband: $S(t) = S_\mathrm{I}(t)\cos(2\pi f_\mathrm{c} t + \Psi) - S_\mathrm{Q}(t)\sin(2\pi f_\mathrm{c} t + \Psi)$
with Ψ uniformly distributed on $[0, 2\pi)$, independent of $\tilde{S}(t)$.

Baseband: $\tilde{S}(t) = S_\mathrm{I}(t) + jS_\mathrm{Q}(t)$

Mean: $m_S = 0$

ACF: $r_S(\tau) = \frac{r_{S_\mathrm{I}}(\tau) + r_{S_\mathrm{Q}}(\tau)}{2}\cos(2\pi f_\mathrm{c}\tau) - \frac{r_{S_\mathrm{I},S_\mathrm{Q}}(-\tau) - r_{S_\mathrm{I},S_\mathrm{Q}}(\tau)}{2}\sin(2\pi f_\mathrm{c}\tau)$

PSDs: $R_S(f) = \frac{1}{4}\big(R_{\tilde{S}}(f-f_\mathrm{c}) + R_{\tilde{S}}(-f-f_\mathrm{c})\big)$, $\quad R_{\tilde{S}}(f) = 4R_S(f+f_\mathrm{c})u(f+f_\mathrm{c})$