TSDT14 Signal Theory

Solutions to the exam 2019-10-23

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1

These subtasks have to be treated correctly as partial fulfillment to pass the exam.

a. A time-discrete stochastic process is an infinite sequence of stochastic variables.

b. The PSD $R_Y(f)$ of the output is given by

$$R_Y(f) = |H(f)|^2 R_X(f) = \begin{cases} 4R_0, & |f| < 1, \\ 0, & \text{elsewhere} \end{cases}$$

c. According to Page 15 in the T & F booklet, we have

$$\begin{aligned} r_Y(\tau) &= 9r_X^2(0)r_X(\tau) + 6r_X^3(\tau) \\ &= 9\,\mathrm{sinc}(\tau) + 6\,\mathrm{sinc}^3(\tau), \end{aligned}$$

where we have used the relation $\mathrm{sinc}(0) = 1$.

2

We have a WSS time-continuous Gaussian process, $X(t)$, with mean zero and ACF

$$r_X(\tau) = R_0\,\mathrm{sinc}^2(\tau/T),$$

which is the input to an ideal BP filter with cutoff frequencies $1/2T$ and $3/2T$, with amplification 1 in the passband.

We want to determine the probability that the output $Y(t)$ is larger than 1.

Using the symmetry relation (second identity in the table on page 18 in T&F), the fact that the ACF is an even function, and the Fourier transform of a triangle (third identity in the table on page 19 in T&F), we write the PSD of $X(t)$ as

$$R_X(f) = \mathcal{F}\{r_X(\tau)\} = R_0 T\,\mathrm{triangle}(fT),$$

This signal is the input to a BP filter with frequency response

$$H(f) = \begin{cases} 1, & \frac{1}{2T} < |f| < \frac{3}{2T}, \\ 0, & \text{elsewhere.} \end{cases}$$

According to the super formula on page 9 in T&F, the output of that filter, $Y(t)$, has PSD

$$R_Y(f) = \big|H(f)\big|^2 R_X(f) = \begin{cases} R_X(f), & \frac{1}{2T} < |f| < \frac{3}{2T}, \\ 0, & \text{elsewhere.} \end{cases}$$

We notice that $R_X(f)$ is zero for $|f| \geq 1/T$. Thus, we can rewrite the output PSD as

$$\begin{aligned} R_Y(f) &= \big|H(f)\big|^2 R_X(f) \\ &= \begin{cases} R_0 T(1 - |f|T), & \frac{1}{2T} < |f| < \frac{1}{T}, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

We observe that we have an LTI system whose input is Gaussian. Thus the output $Y(t)$ is also Gaussian. To determine the probabilty that the output $Y(t)$ is larger than 1, we need the mean and the variance of $Y(t)$. Again, according to page 9 in T&F, we have the output mean

$$m_Y = m_X H(0) = 0,$$

since m_X is zero. Then for the variance, we have

$$\begin{aligned} \sigma_Y^2 &= r_Y(0) - m_Y^2 = r_Y(0) \\ &= \int_{-\infty}^{\infty} R_X(f)\,df = 2 \cdot \frac{1}{2} \cdot \frac{R_0 T}{2} \cdot \left(\frac{1}{T} - \frac{1}{2T}\right) = \frac{R_0}{4}, \end{aligned}$$

and thus we have the output standard deviation

$$\sigma_Y = \frac{\sqrt{R_0}}{2}.$$

We notice that $Y(t)/\sigma_Y$ is $N(0,1)$. The probability is therefore given as

$$\begin{aligned} \Pr\{Y(t) > 1\} &= \Pr\left\{\frac{Y(t)}{\sigma_Y} > \frac{1}{\sigma_Y}\right\} \\ &= Q\left(\frac{1}{\sigma_Y}\right) = Q\left(\frac{2}{\sqrt{R_0}}\right) \end{aligned}$$

3

We have a time-discrete process, $X[n]$, consisting of independent, identically distributed, binary variables, taking values ± 1, and p is the probability that $X[n]$ is 1.

a. We want the mean and ACF of $X[n]$. The mean is given by

$$m_X[n] = p \cdot 1 + (1-p) \cdot (-1) = 2p - 1.$$

The ACF $r_X[n+k, n]$ for $k = 0$ is given by

$$r_X[n, n] = \mathrm{E}\{X^2[n]\} = \mathrm{E}\{1\} = 1.$$

For $k \neq 0$, we find

$$r_X[n+k, n] = \mathrm{E}\{X[n+k]X[n]\} = m_X^2 = (2p-1)^2,$$

where we have used the independence of consequtive samples in the second equality. As we can see, neither the mean $m_X[n]$ nor the ACF $r_X[n+k, n]$ depends on n. We have a WSS process with mean and ACF

$$m_X = 2p - 1,$$
$$r_X[k] = \begin{cases} 1, & k = 0, \\ (2p-1)^2, & k \neq 0. \end{cases}$$

b. Now we want to determine if the process $Y[n] = X[n]X[n-1]$ is WSS. The mean is given by

$$m_Y[n] = \mathrm{E}\{X[n]X[n-1]\} = r_X[1] = (2p-1)^2,$$

which does not depend on n. The ACF $r_Y[n+k, n]$ for $k = 0$ is given by

$$r_Y[n, n] = \mathrm{E}\{X^2[n]X^2[n-1]\} = \mathrm{E}\{1\} = 1.$$

For $k = 1$, we find

$$\begin{aligned} r_Y[n+1, n] &= \mathrm{E}\{X[n+1]X[n]X[n]X[n-1]\} \\ &= \mathrm{E}\{X[n+1]X[n-1]\} \\ &= r_X[2] = (2p-1)^2. \end{aligned}$$

Similarly for $k = -1$, we find

$$r_Y[n-1, n] = r_X[2] = (2p-1)^2.$$

For $k > 1$, we find

$$\begin{aligned} r_Y[n+k, n] &= \mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\} \\ &= m_X^4 = (2p-1)^4, \end{aligned}$$

where we have used the independence of consequtive samples in the second equality. Similarly for $k < -1$, we find

$$r_Y[n+k, n] = m_X^4 = (2p-1)^4,$$

We notice that neither the mean $m_Y[n]$ nor the ACF $r_Y[n+k, n]$ depends on n. Thus, we have a WSS process with mean and ACF

$$m_Y = (2p-1)^2,$$
$$r_Y[k] = \begin{cases} 1, & k = 0, \\ (2p-1)^2, & |k| = 1, \\ (2p-1)^4, & |k| > 1. \end{cases}$$

So yes, the process $Y[n]$ is WSS.

4

We have PAM with pulse-shape $p(t)$. The output is constant $Y[n]$ in an interval of duration T. Identifying in the expression

$$Z(t) = \sum_{n=-\infty}^{\infty} Y[n]p(t - nT - \Psi),$$

where Ψ is uniformly distributed over $[0, T)$, gives us

$$p(t) = \begin{cases} 1, & T_0 \leq t < T + T_0, \\ 0, & \text{elsewhere}, \end{cases}$$

for some constant T_0. The Tables and Formulas booklet, page 18, gives us

$$|P(f)|^2 = T^2 \operatorname{sinc}^2(fT).$$

Let $H[\theta]$ be the frequency response of the filter that generates $Y[n]$ from $X[n]$. This gives us

$$H[\theta] = 1 - e^{-j2\pi\theta} = e^{-j\pi\theta}\left(e^{j\pi\theta} - e^{-j\pi\theta}\right),$$

which in turn gives us

$$|H[\theta]|^2 = \left|e^{j\pi\theta} - e^{-j\pi\theta}\right|^2 = 4\sin^2(\pi\theta) = 2[1 - \cos(2\pi\theta)].$$

We get the PSD of the input by determining the Fourier transform of its ACF:

$$R_X[\theta] = \frac{1}{2}e^{j2\pi\theta} + 1 + \frac{1}{2}e^{-j2\pi\theta} = 1 + \cos(2\pi\theta).$$

LTI-filtering (Tables and Formulas, p. 9) and PAM (Tables and Formulas, p. 12) gives us

$$\begin{aligned} R_Z(f) &= \\ &= \frac{1}{T}|P(f)|^2 \cdot |H[fT]|^2 \cdot R_X[fT] \\ &= T \cdot \operatorname{sinc}^2(fT) \cdot 2[1 - \cos(2\pi fT)][1 + \cos(2\pi fT)] \\ &= T \cdot \operatorname{sinc}^2(fT) \cdot 2[1 - \cos^2(2\pi fT)] \\ &= T \cdot \operatorname{sinc}^2(fT) \cdot 2\sin^2(2\pi fT), \end{aligned}$$

which is our answer.

5

We are given the definition of the cross correlation function (CCF) of two jointly WSS complex processes, $X(t)$ and $Y(t)$,

$$r_{X,Y}(\tau) = \mathrm{E}\{X(t+\tau)Y^*(t)\},$$

and the definion of the pseudo-CCF of those processes,

$$\tilde{r}_{X,Y}(\tau) = \mathrm{E}\{X(t+\tau)Y(t)\}.$$

We want to determine demands on the CCF and/or pseudo-CCF of $X(t)$ and $Y(t)$ to guarantee that

$$Z(t) = X(t) + Y(t)$$

is circular if both $X(t)$ and $Y(t)$ are circular.

Since both $X(t)$ and $Y(t)$ are supposed to be circular, we have

$$\tilde{r}_X(\tau) = \mathrm{E}\big\{X(t+\tau)X(t)\big\} = 0,$$
$$\tilde{r}_Y(\tau) = \mathrm{E}\big\{Y(t+\tau)Y(t)\big\} = 0.$$

We now want to study the pseudo-ACF of $Z(t)$ and determine under what circumstances it is zero. We have

$$\begin{aligned}\tilde{r}_Z(\tau) &= \mathrm{E}\big\{Z(t+\tau)Z(t)\big\} \\ &= \mathrm{E}\big\{\big(X(t+\tau)+Y(t+\tau)\big)\big(X(t)+Y(t)\big)\big\} \\ &= \mathrm{E}\big\{X(t+\tau)X(t)\big\} + \mathrm{E}\big\{X(t+\tau)Y(t)\big\} \\ &\quad + \mathrm{E}\big\{Y(t+\tau)X(t)\big\} + \mathrm{E}\big\{Y(t+\tau)Y(t)\big\} \\ &= \tilde{r}_X(\tau) + \tilde{r}_{X,Y}(\tau) + \tilde{r}_{Y,X}(\tau) + \tilde{r}_Y(\tau) \\ &= \tilde{r}_{X,Y}(\tau) + \tilde{r}_{Y,X}(\tau),\end{aligned}$$

where we have used that $X(t)$ and $Y(t)$ are circular in the last equality. Now, $Z(t)$ is circular if $\tilde{r}_Z(\tau)$ is zero. Thus, we have to demand

$$\tilde{r}_{X,Y}(\tau) = -\tilde{r}_{Y,X}(\tau).$$

Two questions remain:

- Are there pairs of processes for which this relation holds?
- Are there pairs of processes for which this relation does not hold?

The answer to both those questions is yes.

- An example of two jointly WSS circular processes for which the result holds is $Y(t) = X(t)$. Then $\tilde{r}_{X,Y}(\tau)$ and $\tilde{r}_{Y,X}(\tau)$ are both equal to $\tilde{r}_X(\tau)$, which is zero since $X(t)$ is circular, and the relation holds trivially.
- An example of two jointly WSS circular processes for which the result does not hold is $Y(t) = X^*(t)$. Then we have $\tilde{r}_{X,Y}(\tau) = r_X(\tau)$ and $\tilde{r}_{Y,X}(\tau) = r_X^*(\tau)$, which in all interesting cases have a non-zero real part. Thus, the relation does not hold.

In other words, it makes sense to check if the relation holds.

6

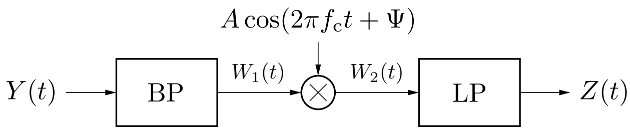
Given situation: $M(t)$ is a WSS strictly bandlimited baseband message with bandwidth W. This message is modulated using AM-SC, i.e. the sent signal is

$$X(t) = M(t)\cos(2\pi f_c t + \Psi),$$

where in turn Ψ is uniformly distributed over $[0, 2\pi)$ and where the carrier frequency fulfills $f_c > W$. The channel adds WGN $N(t)$ which means that the received signal $Y(t)$ is given by

$$Y(t) = X(t) + N(t).$$

$M(t)$, Ψ and $N(t)$ are assumed to be mutually independent. The receiver is given in the figure below, with a non-zero A.



The two filters are ideal, with cut-off frequencies chosen so that the useful signal passes the filter unchanged. We assume full synchronization, i.e. Ψ is completely known at the receiver. Finally, we are supposed to show that the resulting SNR at the output of the receiver does not depend on A.

The short solution is the following, which uses the fact that everything is linear here. Let the output be

$$Z(t) = M_3(t) + N_3(t),$$

where $M_3(t)$ is the useful part of the output, and where $N_3(t)$ is the remaining noise in the output, for the case where we have $A = 1$. The SNR of the output is then given by

$$\mathrm{SNR} = \frac{P_{M_3}}{P_{N_3}}.$$

Now, consider an arbitrary non-zero A. Then the output is given by

$$Z(t) = AM_3(t) + AN_3(t),$$

and the power of the useful part of $Z(t)$ is the power of $AM_3(t)$, which is $A^2 P_{M_3}$. Similarily, the power of the noise in $Z(t)$ is the power of $AN_3(t)$, which is $A^2 P_{N_3}$. The resulting SNR is now

$$\mathrm{SNR} = \frac{A^2 P_{M_3}}{A^2 P_{N_3}} = \frac{P_{M_3}}{P_{N_3}},$$

which does not depend on A. However, for this to make any sense, we have to demand that A is non-zero as the problem formulation states.

Note: An unnecessarily long solution would be to define useful signals and noise for each signal in the figure, i.e.

$$\begin{aligned} W_1(t) &= M_1(t) + N_1(t), \\ W_2(t) &= M_2(t) + N_2(t), \\ Z(t) &= M_3(t) + N_3(t), \end{aligned}$$

where $M_k(t)$ are useful signals and $N_k(t)$ are noise in the three positions, and express signals and PSDs and finally get exact expressions of P_{M_3} and P_{N_3} above, which would both contain A^2 as a factor. However, the essential part of such a solution is actually the short solution given above.