TSDT14 Signal Theory

Solutions to the exam 2020-01-08

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1
These subtasks have to be treated correctly as partial fulfillment to pass the exam.

a. A time-discrete process $X[n]$ is said to be stationary in the wide sense if its ACF $r_X[n_1, n_2]$ only depends on the difference $n_1 - n_2$ and its mean is constant.

b. We have an LTI system. Thus, the PSD of the output $Y(t)$ is given by

$$R_Y(f) = \big|H(f)\big|^2 R_X(f) = \begin{cases} 16R_0, & 1 < |f| < 2, \\ 0, & \text{elsewhere} \end{cases}$$

c. We have a stochastic variable that is uniformly distributed on the interval $[-A, A]$. It is uniformly quantized with $N = 1024$ quantization steps and saturation levels $\pm A$. The usual assumptions are assumed, i.e. no saturation, and small enough Δ compared to the variations of the PDF of the signal. For the quantization step Δ, we have

$$N\Delta = 2A.$$

From that we find the quantization step

$$\Delta = \frac{2A}{N}.$$

The quantization error Q has according to the Tables and Formulas booklet on page 16 the power

$$P_Q = \frac{\Delta^2}{12} = \frac{A^2}{3N^2} = \frac{A^2}{3 \cdot 2^{20}},$$

and that is our distorsion.

2
We have a WSS Gaussian process $X[n]$ with mean m_X and ACF

$$r_X[k] = R_0\delta[k] + m_X^2, \qquad (1)$$

and we are interested in the process $Y[n] = X[n]X[n-1]$.

a. We want to determine the mean and ACF of $Y[n]$. The mean is given by

$$\begin{aligned} m_Y[n] &= \mathrm{E}\big\{Y[n]\big\} = \mathrm{E}\big\{X[n]X[n-1]\big\} \\ &= r_X[1] = m_X^2. \end{aligned}$$

The ACF is given by

$$\begin{aligned} &r_Y[n, n+k] = \\ &= \mathrm{E}\big\{Y[n]Y[n+k]\big\} \\ &= \mathrm{E}\big\{X[n]X[n-1]X[n+k]X[n+k-1]\big\}. \end{aligned}$$

This is the expectation of a product of four jointly Gaussian variables. Therefore, according to page 16 in T & F, this can be rewritten as

$$\begin{aligned} &r_Y[n, n+k] = \\ &= \mathrm{E}\big\{X[n]X[n-1]\big\}\mathrm{E}\big\{X[n+k]X[n+k-1]\big\} \\ &\quad + \mathrm{E}\big\{X[n]X[n+k]\big\}\mathrm{E}\big\{X[n-1]X[n+k-1]\big\} \\ &\quad + \mathrm{E}\big\{X[n]X[n+k-1]\big\}\mathrm{E}\big\{X[n-1]X[n+k]\big\} \\ &\quad - 2\mathrm{E}\big\{X[n]\big\}\mathrm{E}\big\{X[n-1]\big\}\mathrm{E}\big\{X[n+k]\big\}\mathrm{E}\big\{X[n+k-1]\big\} \\ &= r_X^2[1] + r_X^2[k] + r_X[k-1]r_X[k+1] - 2m_X^4. \end{aligned}$$

Using Equation 1, we rewrite the ACF as

$$\begin{aligned} &r_Y[n, n+k] = \\ &= m_X^4 + \big(R_0\delta[k] + m_X^2\big)^2 \\ &\quad + \big(R_0\delta[k-1] + m_X^2\big)\big(R_0\delta[k+1] + m_X^2\big) - 2m_X^4 \\ &= m_X^4 + R_0^2\delta^2[k] + 2m_X^2R_0\delta[k] + m_X^4 \\ &\quad + R_0^2\delta[k-1]\delta[k+1] + R_0m_X^2\big(\delta[k-1] + \delta[k+1]\big) \\ &\quad + m_X^4 - 2m_X^4 \end{aligned}$$

Here, we can observe that

$$\delta^2[k] = \delta[k] \qquad \text{and} \qquad \delta[k-1]\delta[k+1] = 0$$

hold. Using that, we rewrite the ACF as

$$\begin{aligned} r_Y[n, n+k] = &R_0(R_0 + 2m_X^2)\delta[k] \\ &+ R_0m_X^2\big(\delta[k-1] + \delta[k+1]\big) + m_X^4. \end{aligned}$$

b. We notice that the mean $m_Y[n]$ and the ACF $r_Y[n, n+k]$ do not depend on n. Thus, the process $Y[n]$ is WSS.

3

We have the ACF

$$r_X(\tau) = \text{sinc}(\tau/T),$$

which according to T & F, p. 19, gives us the PSD

$$R_X(f) = T \cdot \text{rect}(fT).$$

The pulse-amplitude modulator uses the pulse

$$p(t) = \text{sinc}^2(t/T),$$

which according to T & F, p. 19, gives us the spectrum

$$P(f) = T \cdot \text{triangle}(fT).$$

According to T & F, p. 13, we have the distorsion

$$\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} \left|\frac{P(f)}{T} - 1\right|^2 R_X(f)\, df \\
&\quad + \int_{-\infty}^{\infty} \sum_{n\neq 0} \left|\frac{P(f)}{T}\right|^2 R_X\left(f - \frac{n}{T}\right) df \\
&= 2\int_0^{1/2T} \left|\frac{P(f)}{T} - 1\right|^2 T\, df + 2\int_{1/2T}^{1/T} \left|\frac{P(f)}{T}\right|^2 T\, df \\
&= 2\int_0^{1/2T} \left|\text{triangle}(fT) - 1\right|^2 T\, df \\
&\quad + 2\int_{1/2T}^{1/T} \left|\text{triangle}(fT)\right|^2 T\, df \\
&= 2\int_0^{1/2T} (fT)^2 T\, df + 2\int_{1/2T}^{1/T} (1-fT)^2 T\, df \\
&= 4T^3 \int_0^{1/2T} f^2\, df = 4T^3 \left[\frac{f^3}{3}\right]_0^{1/2T} \\
&= 4T^3 \frac{(1/2T)^3}{3} = \frac{1}{6}.
\end{aligned}$$

The signal power is given by

$$P_X = r_X(0) = 1.$$

That gives us the signal-to-noise ratio

$$\text{SNR} = \frac{P_X}{\epsilon^2} = 6.$$

Expressed in dB that is

$$\text{SNR}_{\text{dB}} = 10 \cdot \log_{10}(6) \approx 7.8 \text{ dB}$$

4

a. The power of the input is given by

$$P_X = r_X(0) = \int_{-\infty}^{\infty} R_X(f)\, df = \int_{-1}^{1} R_0\, df = 2R_0.$$

The power of the output is given by

$$\begin{aligned}
P_Y = r_Y(0) &= \int_{-\infty}^{\infty} R_Y(f)\, df \\
&= \int_{-\infty}^{\infty} \left|H(f)\right|^2 R_X(f)\, df = \int_{-1}^{1} \left|H(f)\right|^2 R_0\, df
\end{aligned}$$

The frequency response of the filter is given by

$$H(f) = \mathcal{F}\{h(t)\} = \frac{a}{1 + j2\pi f},$$

which gives us

$$\left|H(f)\right|^2 = \frac{a^2}{1 + (2\pi f)^2}.$$

We plug that into the expression of P_Y above, and get

$$P_Y = \int_{-1}^{1} R_0 \frac{a^2}{1+(2\pi f)^2} \cdot df = \frac{R_0 a^2}{\pi} \arctan(2\pi)$$

The demand $P_Y = P_X$ gives us

$$2 = \frac{a^2}{\pi} \arctan(2\pi),$$

from which we get

$$a = \sqrt{\frac{2\pi}{\arctan(2\pi)}} \approx 2.11$$

b. The PDF of a Gaussian variable is finite. Especially, $f_{Y(t)}(y)$ is finite in $y = 2$. Thus we have $\Pr\{Y(t) = 2\} = 0$.

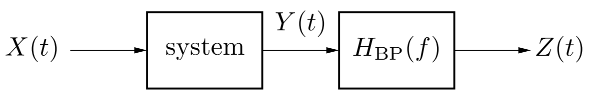
5
The given input process $X(t)$ has ACF

$$r_X(\tau) = \text{sinc}(2B\tau).$$

Then its PSD is given by (T&F, p.19)

$$R_X(f) = \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) = \begin{cases} \frac{1}{2B}, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

We were given the following situation:



The system labelled $H_{\text{BP}}(f)$ is an ideal BP filter with frequency response

$$H_{\text{BP}}(f) = \begin{cases} 1, & B < |f| < 2B, \\ 0, & \text{elsewhere.} \end{cases}$$

We want to determine the output power P_Z for the case where the system labelled *system* is an AM-SC modulator with output

$$Y(t) = A \cdot X(t) \cdot \cos(5\pi B t + \Psi),$$

where Ψ is uniformly distributed over $[0, 2\pi)$ and independent of $X(t)$. according to T&F, p.14, we have

$$\begin{aligned} R_Y(f) &= \frac{A^2}{4}\big(R_X(f+2.5B) + R_X(f+2.5B)\big) \\ &= \begin{cases} \frac{A^2}{8B}, & 1.5B < |f| < 3.5B, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

We get the output power

$$P_Z = \int_{-\infty}^{\infty} \big|H_{\text{BP}}(f)\big|^2 R_Y(f)\,df = 2\int_{1.5B}^{2B} \frac{A^2}{8B}\,df = \frac{A^2}{8}.$$

6
The input can be written as

$$x[n_1,n_2] = \delta[n_1{+}1, n_2{+}1] + \delta[n_1,n_2] + \delta[n_1{-}1, n_2{-}1]$$

and the impulse response of the system can be written as

$$h[n_1,n_2] = \delta[n_1{+}1, n_2{-}1] + 2\delta[n_1,n_2] + \delta[n_1{-}1, n_2{+}1],$$

where $\delta[n_1,n_2]$ is the two-dimensional space-discrete unit impulse.

a. The output $y[n_1,n_2]$ is given by the convolution

$$y[n_1,n_2] = (x \circledast h)[n_1,n_2].$$

We note that

$$\delta[n_1,n_2] = \delta[n_1]\delta[n_2]$$

holds, where $\delta[n]$ is the one-dimensional time-discrete unit impulse. A single two-dimensional impulse is thus separable, and if we convolve two shifted two-dimensional impulses, we get the product of two convolutions of shifted one-dimensional impulses. If we convolve two shifted one-dimensional impulses, the result is a shifted impulse, where the shift is the sum of the two shifts. Therefore, the result of a convolution of two shifted two-dimensional impulses is a shifted two-dimensional impulse, where the shift is the sum of the shifts of the two convolved impulses. Finally, this means that the output of our system is

$$\begin{aligned} &y[n_1,n_2] = \\ &= (x \circledast h)[n_1,n_2] \\ &= \delta[n_1+2,n_2] + 2\delta[n_1+1,n_2+1] + \delta[n_1,n_2+2] \\ &+ \delta[n_1+1,n_2-1] + 2\delta[n_1,n_2] + \delta[n_1-1,n_2+1] \\ &+ \delta[n_1,n_2-2] + 2\delta[n_1-1,n_2-1] + \delta[n_1-2,n_2] \\ &= \begin{cases} 2, & (n_1,n_2) \in \{(0,0), \pm(1,1)\} \\ 1, & (n_1,n_2) \in \{\pm(2,0), \pm(1,-1), \pm(0,2)\} \\ 0, & \text{f.ö.} \end{cases} \end{aligned}$$

b. We Fourier transform $x[n_1,n_2]$ and $h[n_1,n_2]$ and get

$$\begin{aligned} X[\theta_1,\theta_2] &= \sum_{n_1}\sum_{n_2} x[n_1,n_2]e^{-j2\pi(\theta_1 n_1 + \theta_2 n_2)} \\ &= e^{j2\pi(\theta_1+\theta_2)} + 1 + e^{-j2\pi(\theta_1+\theta_2)}, \\ H[\theta_1,\theta_2] &= \sum_{n_1}\sum_{n_2} h[n_1,n_2]e^{-j2\pi(\theta_1 n_1 + \theta_2 n_2)} \\ &= e^{j2\pi(\theta_1-\theta_2)} + 2 + e^{-j2\pi(\theta_1-\theta_2)}. \end{aligned}$$

The spectrum of the output is then given by

$$\begin{aligned} Y[\theta_1,\theta_2] &= X[\theta_1,\theta_2]H[\theta_1,\theta_2] \\ &= \left(e^{j2\pi(\theta_1+\theta_2)} + 1 + e^{-j2\pi(\theta_1+\theta_2)}\right) \\ &\cdot \left(e^{j2\pi(\theta_1-\theta_2)} + 2 + e^{-j2\pi(\theta_1-\theta_2)}\right) \\ &= e^{j2\pi\cdot 2\theta_1} + 2e^{j2\pi\cdot(\theta_1+\theta_2)} + e^{j2\pi\cdot 2\theta_2} \\ &+ e^{j2\pi\cdot(\theta_1-\theta_2)} + 2 + e^{-j2\pi\cdot(\theta_1-\theta_2)} \\ &+ e^{-j2\pi\cdot 2\theta_2} + 2e^{-j2\pi\cdot(\theta_1+\theta_2)} + e^{-j2\pi\cdot 2\theta_1} \end{aligned}$$

After inverse transformation, we get

$$
\begin{aligned}
&y[n_1, n_2] = \\
&\quad = \delta[n_1 + 2, n_2] + 2\delta[n_1 + 1, n_2 + 1] + \delta[n_1, n_2 + 2] \\
&\quad + \delta[n_1 + 1, n_2 - 1] + 2\delta[n_1, n_2] + \delta[n_1 - 1, n_2 + 1] \\
&\quad + \delta[n_1, n_2 - 2] + 2\delta[n_1 - 1, n_2 - 1] + \delta[n_1 - 2, n_2]
\end{aligned}
$$

just as before.