TSDT14 Signal Theory

Lecture 2
Stochastic processes

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(Non-)dependencies of Multi-Dimensional Variables

Mutually Independent:

$$F_{\bar{X}}(\bar{x}) = \prod_{i=0}^{N} F_{X_i}(x_i)$$

$\Rightarrow$

Pairwise Independent:

The variables X_i and X_j are independent for all $i \neq j$.

$\Downarrow$ $\Downarrow$

Mutually Uncorrelated:

$$\mathrm{E}\left\{\prod_{i \in I} X_i\right\} = \prod_{i \in I} \mathrm{E}\{X_i\}$$

For all $I \subset \{1,2,\dots,N\}$

$\Rightarrow$

Pairwise Uncorrelated:

The variables X_i and X_j are uncorrelated for all $i \neq j$.

Jointly Gaussian Variables

$\bar{X} = (X_1, \dots, X_N)$ is called Jointly Gaussian if the following holds:

$$f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2} |\Lambda|^{1/2}} \cdot e^{-\frac{1}{2}(\bar{x}-\bar{m})\Lambda^{-1}(\bar{x}-\bar{m})^T}$$

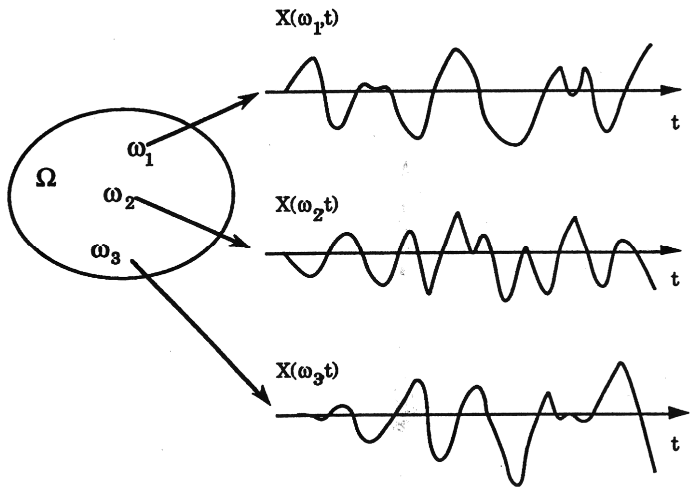
$$\bar{m} = E\{\bar{X}\} \qquad \Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & \ddots & \vdots \\ \lambda_{N1} & \cdots & \lambda_{NN} \end{pmatrix} \qquad \lambda_{ij} = Cov\{X_i, X_j\}$$

If $X_1, \dots, X_N$ are pairwise uncorrelated. $\Rightarrow$ $\Lambda = \begin{pmatrix} \sigma^2_{X_1} & & 0 \\ & \ddots & \\ 0 & & \sigma^2_{X_N} \end{pmatrix}$ $\Rightarrow$

$$\Rightarrow f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2} \cdot \prod_i \sigma_{X_i}} \cdot e^{-\frac{1}{2}\sum_i \frac{(x_i - m_{X_i})^2}{\sigma^2_{X_i}}} = \prod_{i=1}^{N} f_{X_i}(x_i)$$

$\therefore$ Independent

Stochastic Process



Exactly Predictable Process

Definition:

A process is said to be exactly predictable if there exists a finite interval

$$t_1 \le t \le t_2$$

such that it is enough to know a realization in this interval to know the whole realization.

Examples of Stochastic Processes

Ex 1: Finite number of realizations – This one is EPP:

$$X(t) = \sin(t+\Phi), \qquad \Phi \in \{\, 0, \pi/2, \pi, 3\pi/2 \,\}$$

Ex 2: Infinite number of realizations – This one is EPP:

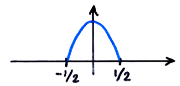
$$X(t) = A\cdot\sin(t), \qquad A \sim \mathrm{N}(0,1)$$

Examples of Stochastic Processes cont'd

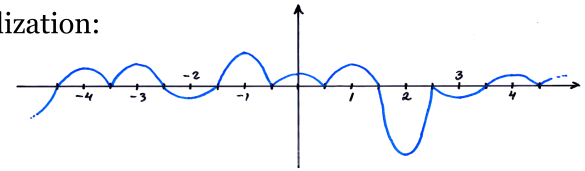
Ex 3: Infinite number of realizations – This one is not EPP:

$$X(t) = \sum A_k \cdot p(t-k), \qquad p(t) = \begin{cases} \cos(\pi t), & |t| < 1/2 \\ 0, & \text{elsewhere} \end{cases}$$

$\{A_k\}$ independent, $\mathrm{N}(0,1)$



A realization:



Distributions and Densities

One time instance

Distribution: $F_{\mathbf{X}(t)}(x) = Pr\{\mathbf{X}(t) \le x\}$

Density: $f_{\mathbf{X}(t)}(x) = \frac{d}{dx} F_{\mathbf{X}(t)}(x)$

Two time instances

Distribution: $F_{\mathbf{X}(t_1),\mathbf{X}(t_2)}(x_1, x_2) = Pr\{\mathbf{X}(t_1) \le x_1, \mathbf{X}(t_2) \le x_2\}$

Density $f_{\mathbf{X}(t_1),\mathbf{X}(t_2)}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{\mathbf{X}(t_1),\mathbf{X}(t_2)}(x_1, x_2)$

Examples of Distributions and Densities

$X(t) = A \cdot \sin(t)$, $\quad$ A is a stochastic variable

$$F_{X(t)}(x) = Pr\{X(t) \le x\} = Pr\{A \cdot \sin(t) \le x\}$$

$$= \begin{cases} Pr\{A \le \frac{x}{\sin(t)}\} = F_A\left(\frac{x}{\sin(t)}\right), & t: \sin(t) > 0 \\ Pr\{0 \le x\} = u(x), & t: \sin(t) = 0 \\ Pr\{A \ge \frac{x}{\sin(t)}\} = 1 - F_A\left(\frac{x}{\sin(t)}\right), & t: \sin(t) < 0 \end{cases}$$

$$f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x) = \begin{cases} \frac{1}{\sin(t)} \cdot f_A\left(\frac{x}{\sin(t)}\right), & t: \sin(t) > 0 \\ \delta(x), & t: \sin(t) = 0 \\ -\frac{1}{\sin(t)} \cdot f_A\left(\frac{x}{\sin(t)}\right), & t: \sin(t) < 0 \end{cases}$$

$$= \begin{cases} \delta(x), & t = k \cdot \pi, \quad k \in \mathbb{Z} \\ \frac{1}{|\sin(t)|} \cdot f_A\left(\frac{x}{\sin(t)}\right), & \text{elsewhere} \end{cases}$$

Multiple Time Instances

Vector notation: $\bar{t} = (t_1, t_2, \dots, t_N)$

$$X(\bar{t}) = (X(t_1), X(t_2), \dots, X(t_N))$$

$$\bar{x} = (x_1, x_2, \dots, x_N)$$

Distribution: $F_{X(\bar{t})}(\bar{x}) = Pr\{X(t_1) \le x_1, X(t_2) \le x_2, \dots, X(t_N) \le x_N\}$

Density: $f_{X(\bar{t})}(\bar{x}) = \frac{\partial^N}{\partial x_1 \partial x_2 \cdots \partial x_N} F_{X(\bar{t})}(\bar{x})$

Ensemble Average

- Expectation of a function of samples of a stochastic process

Mean: $m_X(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx$

Quadratic mean: $E\{X^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x)\, dx$

Variance: $\sigma_X^2(t) = E\{(X(t) - m_X(t))^2\} = E\{X^2(t)\} - m_X^2(t)$

Std deviation: $\sigma_X(t)$

Functions of time!

Auto-correlation function (ACF):

$$r_X(t_1, t_2) = E\{X(t_1) X(t_2)\} = \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

Symmetry: $r_X(t_1, t_2) = r_X(t_2, t_1)$

Power: $r_X(t, t) = E\{X^2(t)\}$

Special case: $X(t)$ fcn of stoch var A:

$$X(t) = g(t, A)$$

$$r_X(t_1, t_2) = \int_{-\infty}^{\infty} g(t_1, a) g(t_2, a) f_A(a)\, da$$

Example of Ensemble Averages

$X(t) = A \sin(t)$, $\quad$ A is a stochastic variable

$$m_X(t) = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx = \int_{-\infty}^{\infty} a \cdot \sin(t) \cdot f_A(a)\, da = \sin(t) \cdot \int_{-\infty}^{\infty} a f_A(a)\, da = \sin(t) \cdot m_A$$

$$E\{X^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x)\, dx = \int_{-\infty}^{\infty} (a \cdot \sin(t))^2 \cdot f_A(a)\, da = \sin^2(t) \int_{-\infty}^{\infty} a^2 f_A(a)\, da = \sin^2(t) \cdot E\{A^2\}$$

$$\sigma_X^2(t) = E\{X^2(t)\} - m_X^2(t) = \sin^2(t)\left(E\{A^2\} - m_A^2\right) = \sin^2(t) \cdot \sigma_A^2$$

$$r_X(t_1, t_2) = \int_{-\infty}^{\infty} a \cdot \sin(t_1) \cdot a \cdot \sin(t_2) \cdot f_A(a)\, da$$

$$= \sin(t_1) \cdot \sin(t_2) \cdot \int_{-\infty}^{\infty} a^2 f_A(a)\, da$$

$$= \sin(t_1) \cdot \sin(t_2) \cdot E\{A^2\}$$

Auto-Correlation and Auto-Covariance

Auto-correlation (ACF):

$$r_X(t_1, t_2) \triangleq \mathrm{E}\{X(t_1)X(t_2)\} = \iint_{-\infty}^{\infty} x_1 x_2 \, f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1\, dx_2,$$

$$r_X(t_1, t_2) = r_X(t_2, t_1)$$

Auto-covariance (related concept):

$$\lambda_X(t_1, t_2) \triangleq \mathrm{E}\Big\{\big(X(t_1) - m_X(t_1)\big)\big(X(t_2) - m_X(t_2)\big)\Big\}$$

$$\lambda_X(t_1, t_2) = r_X(t_1, t_2) - m_X(t_1)m_X(t_2).$$

Stationarity 1(2)

Stationarity is statistical invariance to a shift of the time origin.

Definition:

Consider time instances $\overline{t} = (t_1,\ldots,t_N)$ and shifted time instances $\overline{u} = \overline{t} + \Delta = (t_1+\Delta,\ldots,t_N+\Delta)$. The process $X(t)$ is said to be <u>stationary in the strict sense</u> (SSS) if

$$F_{X(\overline{t})}(\overline{x}) = F_{X(\overline{u})}(\overline{x})$$

holds for all N and all choices of $\overline{t}$ and Δ.

Equivalence:

$$F_{X(\overline{t})}(\overline{x}) = F_{X(\overline{u})}(\overline{x}) \quad \Leftrightarrow \quad f_{X(\overline{t})}(\overline{x}) = f_{X(\overline{u})}(\overline{x})$$

Stationarity 2(2)

Mean: $m_X(t) = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx \overset{\text{stat.}}{=} \int_{-\infty}^{\infty} x f_{X(t+\Delta)}(x)\, dx = m(t+\Delta) \quad \forall \Delta$

Thus Constant.

ACF:

$$r_X(t_1, t_2) = \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

$$= \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1+\Delta),X(t_2+\Delta)}(x_1, x_2)\, dx_1\, dx_2$$

$$= r_X(t_1+\Delta, t_2+\Delta)$$

Dep. on $t_1 - t_2$

Notation: m_X and $r_X(\tau)$ $\qquad \tau = t_1 - t_2$

Wide sense stationarity, definition:

If the above holds, then the process $X(t)$ is said to be <u>stationary in the wide sense</u> (WSS).

Gaussian Processes

Recall: $\overline{X} = (X_1,\ldots,X_N)$ is called Jointly Gaussian if the following holds:

$$f_{\overline{X}}(\overline{x}) = \frac{1}{(2\pi)^{N/2}|\Lambda|^{1/2}} \cdot e^{-\frac{1}{2}(\overline{x}-\overline{m})\Lambda^{-1}(\overline{x}-\overline{m})^T}$$

$$\overline{m} = E\{\overline{X}\} \qquad \Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & \ddots & \vdots \\ \lambda_{N1} & \cdots & \lambda_{NN} \end{pmatrix} \qquad \lambda_{ij} = Cov\{X_i, X_j\}$$

Definition: A stochastic process is called *Gaussian* if all its multidimensional PDFs correspond to jointly Gaussian variables.

Theorem: A Gaussian process that is stationary in the wide sense is also stationary in the strict sense.

Power-Spectral Density (PSD)

Definition: Fourier transform of the ACF:

$$R_X(f) = \mathcal{F}\{r_X(\tau)\} = \int_{-\infty}^{\infty} r_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

Inverse:

$$r_X(\tau) = \mathcal{F}^{-1}\{R_X(f)\} = \int_{-\infty}^{\infty} R_X(f)\, e^{j2\pi f\tau}\, df$$

Power:

$$E\{X^2(t)\} = r_X(0) = \int_{-\infty}^{\infty} R_X(f)\, df$$

Ergodicity 1(2)

A WSS process: $m_X = E\{X(t)\}$

Time-average of one realization: $m_T = \frac{1}{2T}\int_{-T}^{T} x(t)\, dt$

Time-average of the process: $M_T = \frac{1}{2T}\int_{-T}^{T} X(t)\, dt$

$$E\{M_T\} = E\left\{\frac{1}{2T}\int_{-T}^{T} X(t)\, dt\right\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\, dt = \frac{1}{2T}\int_{-T}^{T} m_X\, dt = m_X$$

Definition: If $\lim_{T\to\infty} E\{(M_T - m_X)^2\} = 0$ then $X(t)$ is said to be ergodic with respect to the mean, and we write

$$m_X = \underset{T\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2T}\int_{-T}^{T} X(t)\, dt \qquad \text{(Limes in mean square)}$$

Interpretation: The time-average of a process is very close to the ensemble mean with probability that is very close to 1 ($\to 1, T \to \infty$).

Ergodicity 2(2)

Definition: A process that is ergodic with respect to all ensemble averages is simply said to be ergodic.

Theorem: An ergodic process is SSS.

Theorem: A SSS Gaussian process with mean zero is ergodic if and only if its PSD has no impulses.

Theorem: A SSS process, $X(t)$, is ergodic with respect to its mean if and only if

$$\underset{T\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2T}\int_{-T}^{T} r_X(\tau)\, d\tau = m_X^2$$

holds.

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