TSDT14 Signal Theory

Solutions to the exam 2018-08-29

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1

At least two of the following three sub-tasks have to be treated correctly as partial fulfillment to pass the exam:

a. The definition that we have been using in this course is that the PSD of the process is constant, which among other things assumes that the process is stationary in the wide sense. There is an alternative definition which demands that different samples are uncorrelated, which corresponds to a constant PSD for non-zero frequencies and that there may be an impulse in the origin, which is caused by the mean of the process.

b. We were given the PSD

$$R_X[\theta] = 1 + \cos(2\pi\theta),$$

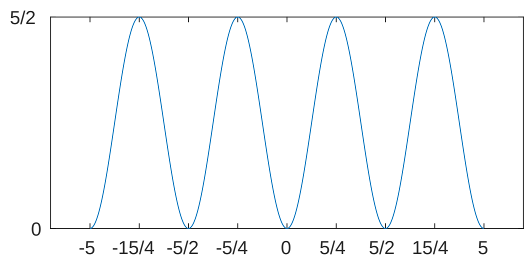
and the spectrum

$$P(f) = \begin{cases} j \cdot \sin(2\pi f/10), & |f| < 5 \text{ Hz}, \\ 0, & \text{elsewhere} \end{cases}$$

of the pulse, and also the sampling frequency 5 Hz. Then we immediately have

$$\begin{aligned} R_Y(f) &= \frac{1}{T}|P(f)|^2 \cdot R_X[fT] \\ &= \begin{cases} 5\sin^2(2\pi f/10)(1+\cos(2\pi f/5)), & |f| < 5, \\ 0, & \text{elsewhere} \end{cases} \\ &= \begin{cases} \frac{5}{4}(1-\cos(4\pi f/5)), & |f| < 5, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

Graphically:



c. According to Page 15 in the T & F booklet, we have

$$\begin{aligned} r_Y(\tau) &= 9r_X^2(0)r_X(\tau) + 6r_X^3(\tau) \\ &= 9e^{-|\tau|} + 6e^{-3|\tau|}, \end{aligned}$$

where we have used the relation $e^0 = 1$.

2

The signal $Y[n]$ is the output of a filter with impulse response

$$h[n] = \frac{1}{4}\big(\delta[n+1] + 2\delta[n] + \delta[n-1]\big)$$

where the input is $X[n]$. The input is WSS since we express its ACF in only one time-variable. Then $Y[n]$ is also WSS. The frequency response of the filter is given by

$$\begin{aligned} H[\theta] &= \frac{1}{4}\left(e^{j2\pi\theta} + 2 + e^{-j2\pi\theta}\right) = \frac{1}{4}\big(2 + 2\cos(2\pi\theta)\big) \\ &= \frac{1+\cos(2\pi\theta)}{2}. \end{aligned}$$

Using the T & F booklet, we find the PSD

$$R_X[\theta] = \mathcal{F}\{r_X[k]\} = \frac{3}{5 - 4\cos(2\pi\theta)}.$$

The PSD of the output is given by

$$R_Y[\theta] = |H[\theta]|^2 R_X[\theta] = \frac{3\big(1+\cos(2\pi\theta)\big)^2}{4\big(5-4\cos(2\pi\theta)\big)},$$

and its ACF is given directly by the definition

$$\begin{aligned} r_Y[k] &= \\ &= \mathrm{E}\big\{Y[n]Y[n+k]\big\} \\ &= \frac{1}{16}\mathrm{E}\Big\{\big((X[n+1] + 2X[n] + X[n-1]\big)\times \\ &\quad \times \big(X[n+k+1] + 2X[n+k] + X[n+k-1]\big)\Big\} \\ &= \frac{1}{16}\big(r_X[k-2] + 4r_X[k-1] + 6r_X[k] + \\ &\quad + 4r_X[k+1] + r_X[k+2]\big) \\ &= \frac{1}{16}\Big(2^{-|k-2|} + 4\cdot 2^{-|k-1|} + 6\cdot 2^{-|k|} + \\ &\quad + 4\cdot 2^{-|k+1|} + 2^{-|k+2|}\Big). \end{aligned}$$

The power of $Y[n]$ is

$$E\{Y^2[n]\} = r_Y[0] = \frac{21}{32} = 0.65625.$$

3

We have $Y(t) = x(t+\Psi)$, where Ψ is a stochastic variable, uniformly distributed on $[0, T)$, and where we have the periodic function

$$x(t) = \begin{cases} 1, & 0 < t \le T/2 \\ 0, & T/2 < t \le T \end{cases}$$

and $x(t) = x(t + nT)$, where n is an integer and T is its period.

a. To determine if $Y(t)$ is WSS, we need to determine two things, namely that the mean $m_Y(t)$ and the ACF $r_Y(t, t+\tau)$ are both independent of t.

Let us start with the mean:

$$m_Y(t) = \mathrm{E}\{Y(t)\} = \int_0^T x(t+\psi) f_\Psi(\psi)\, d\psi$$

We integrate over one full period of the signal and $f_\Psi(\psi)$ is constant over the complete integration interval. Thus, we can integrate over any full period of the signal, and we get

$$m_Y(t) = \int_0^T x(\psi) f_\Psi(\psi)\, d\psi,$$

which does not depend on t.

Then the ACF:

$$\begin{aligned} r_Y(t, t+\tau) &= \mathrm{E}\{Y(t)Y(t+\tau)\} \\ &= \int_0^T x(t+\psi)x(t+\tau+\psi) f_\Psi(\psi)\, d\psi. \end{aligned}$$

Again, we integrate over a full period of the signal and $f_\Psi(\psi)$ is constant over the complete integration interval. Thus, we can again integrate over any full period of the signal, and we get

$$r_Y(t, t+\tau) = \int_0^T x(\psi)x(\tau+\psi) f_\Psi(\psi)\, d\psi$$

whis is also independent of t.

To conclude, neither $m_Y(t)$ nor $r_Y(t, t+\tau)$ depend on t. $Y(t)$ is thus WSS.

b. We define as usual the stochastic variable

$$M_{T_0} = \frac{1}{2T_0}\int_{-T_0}^{T_0} Y(t)\, dt$$

The signal is ergodisk with respect to its mean if $E\left\{(M_T - m_Y)^2\right\}$ tends to zero as T_0 tends to ∞. Let us therefore study the quadratic mean

$$\begin{aligned} \mathrm{E}\{M_{T_0}^2\} &= \mathrm{E}\left\{\left(\frac{1}{2T_0}\int_{-T_0}^{T_0} Y(t)\, dt\right)^2\right\} \\ &= \frac{1}{4T_0^2}\int_{-T_0}^{T_0}\int_{-T_0}^{T_0} \mathrm{E}\{x(t+\Psi)x(\tau+\Psi)\}\, dt\, d\tau \\ &= \frac{1}{4T_0^2}\int_{-T_0}^{T_0}\int_{-T_0}^{T_0} r_Y(t-\tau)\, dt\, d\tau. \end{aligned}$$

The ACF of $Y(t)$ that we determined above can be rewritten as

$$r_Y(\tau) = \begin{cases} \frac{1}{2} - \frac{\tau}{T}, & 0 < \tau \le T/2, \\ \frac{\tau}{T} - \frac{1}{2}, & T/2 < \tau \le T, \end{cases}$$

repeated with period T. The ACF $r_Y(\tau)$ is thus a periodic function with period T, and given the symmetry of the ACF, we also have

$$\int_t^{t+T} r_Y(\tau)\, d\tau = T/4.$$

This gives us the possibility to bound the inner integral in the expression of the quadratic mean, and we get

$$\begin{aligned} \int_{-T_0}^{T_0} \frac{2T_0 - T}{16T_0^2}\, d\tau \le \mathrm{E}\{M_{T_0}^2\} &\le \int_{-T_0}^{T_0} \frac{2T_0 + T}{16T_0^2}\, d\tau \\ 2T_0\frac{2T_0 - T}{16T_0^2} \le \mathrm{E}\{M_{T_0}^2\} &\le 2T_0\frac{2T_0 + T}{16T_0^2} \\ \frac{1}{4} - \frac{T}{8T_0} \le \mathrm{E}\{M_{T_0}^2\} &\le \frac{1}{4} + \frac{T}{8T_0} \end{aligned}$$

Both the upper and lower bound tends to $1/4$ as T_0 tends to ∞. Hence, the quadratic mean also tends to $1/4$. The mean is trivially given as $m_Y = 1/2$, and hence the variance is given by

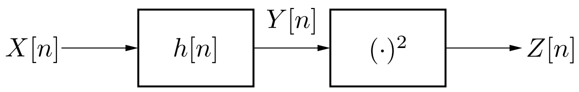
$$\sigma_{M_{T_0}}^2 = \mathrm{E}\{M_{T_0}^2\} - m_Y^2 \to \frac{1}{4} - \frac{1}{4} = 0, \quad T_0 \to \infty$$

and thus, $Y(t)$ is ergodic with respect to its mean.

Answer: $Y(t)$ is both WSS and ergodic with respect to its mean..

4

We are given the system below, where the filter has impulse response $h[n] = \delta[n] + \delta[n-1]$.



The time-discrete stochastic signal $X[n]$ is white noise with mean $m_X = 0$ and PSD $R_X[\theta] = 1$.

The ACF of $X[n]$ is the inverse transform of $R_X[\theta] = 1$. Thus, we have $r_X[k] = \delta[k]$. In both subtasks, we are interested in the mean of the signal $Y[n]$. It is given by

$$m_Y = H[0]m_X = 0,$$

where $H[\theta]$ is the frequency response of the filter.

a. We want to determine the ACF $r_Z[k]$ if $X[n]$ is Gaussian. Then we can use the relation given on page 15 in T& F. Therefore, we are first interested in the ACF

$$r_Y[k] = (h * \tilde{h} * r_X)[k],$$

where we have $\tilde{h}[n] = h[-n] = \delta[n] + \delta[n+1]$. To start with, we determine the convolution

$$(h * \tilde{h})[k] = \delta[k+1] + 2\delta[k] + \delta[k-1].$$

Using that in the expression of $r_Y[k]$ above, we find

$$\begin{aligned} r_Y[k] &= r_X[k+1] + 2r_X[k] + r_X[k-1] \\ &= \delta[k+1] + 2\delta[k] + \delta[k-1]. \end{aligned}$$

Since $Y[n]$ is Gaussian with mean zero, we can use the relation

$$\begin{aligned} r_Z[k] &= 2r_Y^2[k] + r_Y^2[0] \\ &= 4 + 2\delta[k+1] + 8\delta[k] + 2\delta[k-1]. \end{aligned}$$

b. Now we want to determine the ACF $r_Z[k]$ if $X[n]$ is binary distributed, taking values ± 1. We have no special expression for that, which means that we need to use definitions. First, we have

$$Y[n] = (h * X)[n] = X[n] + X[n-1].$$

Then the output is given by

$$\begin{aligned} Z[n] &= Y^2[n] = \big(X[n] + X[n-1]\big)^2 \\ &= 2 + 2X[n]X[n-1] \end{aligned}$$

The ACF of the output is given by

$$\begin{aligned} r_Z[k] &= \mathrm{E}\{Z[n+k]Z[n]\} = \\ &= \mathrm{E}\Big\{\big(2 + 2X[n+k]X[n+k-1]\big)\cdot \\ &\qquad \cdot \big(2 + 2X[n]X[n-1]\big)\Big\} \\ &= \mathrm{E}\{4\} + \mathrm{E}\{4X[n+k]X[n+k-1]\} \\ &\quad + \mathrm{E}\{4X[n]X[n-1]\} \\ &\quad + \mathrm{E}\{4X[n+k]X[n+k-1]X[n]X[n-1]\} \\ &= 4 + 8r_X[1] \\ &\quad + \mathrm{E}\{4X[n+k]X[n+k-1]X[n]X[n-1]\} \\ &= 4 + 4\mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\}. \end{aligned}$$

Here, we need to observe that binary uncorrelated variables are independent. Since $X[n]$ is white, it consists of uncorrelated variables. Therefore, since $X[n]$ is also binary, it consists of independent samples. Thus, any expectation of products of different samples of this process can be written as the corresponding product of expectations.

For the expectation in our last expression above, we need to study a few cases. For the case $k = 0$, we have

$$\begin{aligned} &\mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\} = \\ &= \mathrm{E}\{X^2[n]X^2[n-1]\} = \mathrm{E}\{1\} = 1. \end{aligned}$$

For the case $|k| = 1$, we have

$$\begin{aligned} &\mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\} = \\ &= \mathrm{E}\{X[n+1]X^2[n]X[n-1]\} \\ &= \mathrm{E}\{X[n+1]X[n-1]\} = m_X^2 = 0. \end{aligned}$$

For the case $|k| > 1$, we have

$$\mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\} = m_X^4 = 0.$$

Combining all of this, we have

$$\mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\} = \delta[k],$$

and finally

$$\begin{aligned} r_Z[k] &= 4 + 4\mathrm{E}\{X[n+k]X[n+k-1]X[n]X[n-1]\} \\ &= 4 + 4\delta[k]. \end{aligned}$$

Answer:

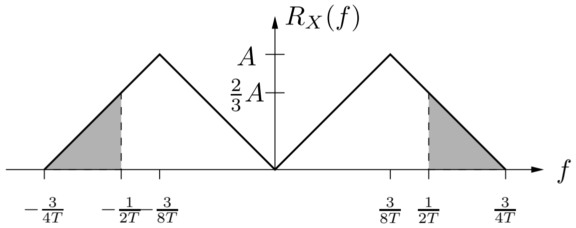
a. $r_Z[k] = 4 + 2\delta[k+1] + 8\delta[k] + 2\delta[k-1]$

b. $r_Z[k] = 4 + 4\delta[k]$

5

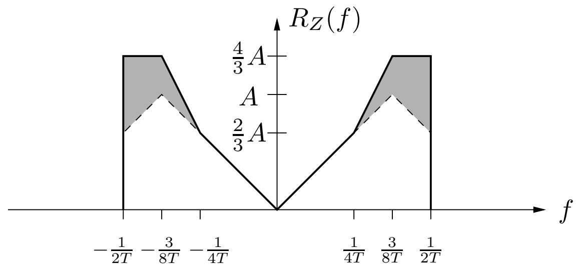
For full credit on this task, you have to explicitly state the used equations. Apart from that, a graphical reasoning is OK.

We have the following PSD, where we have introduced the notation A for the height of the triangles:



The gray parts of the PSD are the parts of the PSD, for which we have $|f| > \frac{1}{2T}$. Let $Y[n]$ denote the intermediate time-discrete signal, and let $Z(t)$ denote the output of the PAM.

a. The PSD of the output is given by folding in the gray parts.



We are here using Poissons summation formula,

$$R_Y[\theta] = f_s \sum_m R_X\big((\theta - m) f_s\big),$$

for sampling (T&F, Page 11) and the relation

$$R_Z(f) = f_s \big|P(f)\big|^2 R_Y[f/f_s]$$

for PAM (T&F, Page 12), resulting in

$$R_Z(f) = \\ = \begin{cases} R_X(f+f_s)+R_X(f)+R_X(f-f_s), & |f|<f_s/2, \\ 0, & \text{elsewhere.} \end{cases}$$

b. The smallest distorsion is achieved if we are using an ideal anti-aliasing filter with exactly half the sampling frequency as cut-off frequency. Then the sampling followed by ideal reconstruction does not introduce any error. The only error is from the filter. The power of that error is the area of the gray parts, since the reconstruction is ideal,

$$\epsilon^2 = 2 \cdot \frac{1}{2} \cdot \frac{2}{3} A \cdot \frac{1}{4T} = \frac{A}{6T}.$$

Answer:
a. See the second figure above.
b. $\epsilon^2 = \frac{A}{6T}$.

6

Let $x(a_1, a_2)$ be our signal. Assuming that it is separable, there are one-dimensional signals $x_1(a)$ and $x_2(a)$, such that we have

$$x(a_1, a_2) = x_1(a_1) \cdot x_2(a_2)$$

holds. Then we have the Fourier transform

$$\begin{aligned} X(f_1, f_2) &= \iint_{-\infty}^{\infty} x(a_1, a_2)\, e^{-j2\pi(f_1 a_1 + f_2 a_2)}\, da_1\, da_2 \\ &= \iint_{-\infty}^{\infty} x_1(a_1)\, e^{-j2f_1 a_1} \cdot x_2(a_2) e^{-j2f_2 a_2}\, da_1\, da_2 \\ &= \int_{-\infty}^{\infty} x_1(a_1)\, e^{-j2f_1 a_1}\, da_1 \cdot \int_{-\infty}^{\infty} x_2(a_2)\, e^{-j2f_2 a_2}\, da_2 \\ &= X_1(f_1) \cdot X_2(f_2). \end{aligned}$$

Conversely, assuming that the Fourier transform is separable, i.e. that there are one-dimensional functions $X_1(f)$ and $X_2(f)$, such that we have

$$X(f_1, f_2) = X_1(f_1) \cdot X_2(f_2)$$

holds. Then doing the same thing with the inverse transform yields

$$\begin{aligned} x(a_1, a_2) &= \iint_{-\infty}^{\infty} X(f_1, f_2)\, e^{j2\pi(f_1 a_1 + f_2 a_2)}\, df_1\, df_2 \\ &= \iint_{-\infty}^{\infty} X_1(f_1)\, e^{j2f_1 a_1} \cdot X_2(f_2) e^{j2f_2 a_2}\, df_1\, df_2 \\ &= \int_{-\infty}^{\infty} X_1(f_1)\, e^{j2f_1 a_1}\, df_1 \cdot \int_{-\infty}^{\infty} X_2(f_2)\, e^{j2f_2 a_2}\, df_2 \\ &= x_1(a_1) \cdot x_2(a_2). \end{aligned}$$

This proof is for space-continuous signals. The proof for space-discrete signals is along the same lines.